Proposition 1. Suppose that $(M, \omega)$ is a symplectic manifold equipped with a compatible almost complex structure $J$. Then there exists a constant $h > 0$ such that $E(u) \geq h$ for every nonconstant $J$-holomorphic sphere $u : S^2 \to M$.

The proof of Proposition 1 relies on the mean value inequality; it is reasonably straightforward once this inequality is established. This proposition is important because it limits the number of bubbles that can appear.

Remark 2. In the $\tau$-monotone case, we can take $h = 1/\tau$. A similar lower bound for energy can be found for holomorphic disks with Lagrangian boundary conditions; see McDuff-Salamon, chapter 4.

Today we prove the weakest notion of convergence for cylinders between periodic orbits. It says essentially that if a sequence has uniformly bounded energy then it converges; otherwise, it converges up to the bubbling off of spheres.

Theorem 3. Let $u_i \in \mathcal{M}(x^\pm, J, H)$ be a sequence of solutions to the Floer equation with uniformly bounded energy. Then there exist finitely many points $z_j \in \mathbb{R} \times S^1$, $j = 1, \ldots, l$, and a connecting trajectory $u$ such that the sequence converges to $u$, uniformly on compact subsets of $\mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_l\}$. Moreover, if we define $h$ as in Proposition 1, then we have

$$E(u) \leq \liminf_{i \to \infty} E(u_i) - l h.$$

Proof. We sketch the proof here; details are non-trivial.

Step 1: Convergent Case. If $|\partial u_i/\partial s|$ is uniformly bounded, then convergence follows from Arzelà-Ascoli and elliptic bootstrapping, with $l = 0$. The elliptic bootstrapping argument is non-trivial, but it guarantees that the limit will be smooth.

Step 2: Find the bubbles. Suppose there exist $\xi_i \in \mathbb{R} \times S^1$ with $|\partial u_i/\partial s(\xi_i)| \to \infty$. It follows from exponential convergence that this sequence is contained in some compact set, so by passing to a subsequence we can assume $\xi_i \to \xi$. Then we can define a sequence of special functions $v_i : \Omega_i \to M$, where $\Omega_1 \subset \Omega_2 \subset \ldots$ is an exhaustion of $\mathbb{C}$ by disks. We should be able to choose $v_i = u_i \circ \varphi_i$, where $\varphi_i : \Omega_i \to U$ is a holomorphic inclusion of $\Omega_i$ into some fixed neighborhood $U$ of $\xi$ satisfying $\varphi_i(0) = \xi_i$. 

\[\begin{array}{c}
\begin{tikzpicture}
\fill[blue, opacity=0.1] (0,0) circle (1);
\fill[red, opacity=0.1] (0,0) circle (0.8);
\fill[green, opacity=0.1] (0,0) circle (0.6);
\fill[black, fill opacity=0.5] (0,0) circle (0.4);
\fill[black, fill opacity=0.5] (0,0) circle (0.2);
\end{tikzpicture}
\end{array}\]
If we pick these functions very carefully, we should be able to guarantee $|dv_i(0)| \geq \frac{1}{2}$ (so the limit will be non-constant) and $|dv_i|_{L^\infty(\Omega_i)} \leq 2$ (equicontinuity). Then by applying Arzelà-Ascoli, we see that the $v_i$ must $C_\infty^{\infty}$-converge to some $v : \mathbb{C} \to M$. This map must have finite energy (energy depends only on the conformal structure, and the $u_i$ have uniformly bounded energy), so we can remove the singularity at $\infty$ to obtain a map $\tilde{v} : S^2 \to M$.

**Step 3: Finitely many bubbles.** By conformal invariance of energy, if $U$ is any neighborhood of a point $\xi$ where energy accumulates (as in step 2), we must have

$$\liminf_{i \to \infty} E(u_i|_U) \geq \hbar.$$  

Thus we can repeat step 2 until we’ve found all the accumulation points, and since $E(u_i)$ is uniformly bounded there can be only finitely many. Label these points $z_1, \ldots, z_l$.

**Step 4: Local convergence away from bubble points.** Away from $z_1, \ldots, z_l$, we now know that $|\partial u_i/\partial s|$ is uniformly bounded, so we can apply Arzelà-Ascoli on a sequence of compact domains to define $u$ on $\mathbb{R} \times S^1 \setminus \{z_1, \ldots, z_l\}$. We get uniform convergence on each of these domains, which yields $C_\infty^{\infty}$-convergence away from $z_1, \ldots, z_l$. Finally, we can use the removal of singularities to define $u$ on all of $\mathbb{R} \times S^1$. □

If we’re more careful about parametrization, we can prove the existence of a broken trajectory to which a subsequence must $C_\infty^{\infty}$-convergence, up to bubbles.

**Remark 4.** In Morse theory, we can show that a glued broken trajectory is homologous to $u_k$ for large $k$. We have to be more careful in Floer theory because we have to worry about the homology of the bubbles. In particular, we haven’t yet discussed bubble trees (bubbles which come out of other bubbles), so we need to do a bit more work and add a gluing statement to make sense of this.

**Theorem 5.** Assume that $M$ is $\tau$-monotone (which guarantees transversality) and that $(H, J)$ is a regular pair. Suppose that $\mathcal{M} = \mathcal{M}(x^\pm, J, H)$ has virtual (and hence actual) dimension one. Then $\hat{\mathcal{M}} = \mathcal{M}/\mathbb{R}$ is a compact 0-manifold.

**Proof.** We know from our computation of dimension that $1 = \eta(x^-) - \eta(x^+) \pm 2\tau E(u)$, so we get a uniform energy bound on $\mathcal{M}$. Hence any sequence in $\mathcal{M}$ has a subsequence which converges to a broken trajectory, up to bubbles.
We need to eliminate both breaking and bubbles. We’ll use type of dimension argument. First, it is a key point that the dimension of broken trajectories passing through periodic orbits $x_0, \ldots, x_n$ is

$$\dim \mathcal{M}(x_0, x_1) + \ldots + \dim \mathcal{M}(x_{n-1}, x_n).$$

This follows from a gluing argument (which shows that a broken trajectory is a limit of trajectories in $\mathcal{M}(x_0, x_n)$) and the mixed index formula. Hence if we start with $\dim \mathcal{M}(x^\pm) = 1$, we can’t see any breaking.

The argument to eliminate bubbles is somewhat more subtle. We must be more precise about what it means to converge “up to bubbles,” but the essential point is that bubbles occur in codimension at least two in the monotone case. We can use a “soft rescaling” argument (a weaker version of gluing) to prove this claim. Suppose that a sequence $u_j$ in a moduli space of dimension $k$ converges to some trajectory $v$ with a bubble (or bubble tree) of Chern number $c$. Then the dimension of such trajectories $v$ is $k - 2c$.

In particular, if $M$ is monontone then $c \geq 1$. In our case we have $k = 1$, so this can’t possibly happen (i.e., codimension two bubbles can’t appear in a dimension one space). It follows that we cannot see bubbles here, so $\hat{\mathcal{M}}$ is compact.

We now have compact moduli spaces when the virtual dimension $\eta(x^-) - \eta(x^+) \pm 2\tau E(u)$ is one, which will allow us to define a sequence of groups $C_k$ along with a differential $d$. However, to check $d^2 = 0$, we’ll need to worry about what happens when the virtual dimension is two. We do expect to see breaking in this case (but if we orient everything correctly, this breaking should happen symmetrically, and it will all cancel). Unfortunately, we don’t have a way to eliminate bubbling a priori. If we start in dimension two, we might degenerate to something that happens in dimension zero (i.e., a periodic orbit) plus a bubble of Chern number one.

To get rid of this possibility, we’ll use the fact that $H$ and $J$ are “generic.” The points in $\mathcal{M}$ traced out by holomorphic spheres with Chern number one have codimension two, so generically they shouldn’t intersect periodic orbits.