We’ve shown that \( D \) is invertible in the special case \( h_t = h \) is constant, and we’ve shown that \( D \) is Fredholm in general. We still need to check that the index of \( D \) is equal to the spectral flow of \( L_0 + h_t \). We’ll only deal with two special cases.

**Case 1:** \( h_t \) is hyperbolic for all \( t \). By definition, the eigenvalues of \( h_t \) never cross \( i\mathbb{R} \), so the spectral flow is zero. To check \( \text{ind}(D) = 0 \), we deform \( D \) to something invertible. We define a family \( D_s \) so that \( D_0 = \frac{d}{dt} + L_0 + h_- \) and \( D_1 = D \).

\[
\begin{array}{c|c|c}
L_0 + h_- & L_0 + h_t & L_0 + h_{t_0} \\
\hline
L_0 + h_- & L_0 + h_t & L_0 + h_{t_0} \\
\hline
L_0 + h_- & L_0 + h_t & L_0 + h_+ \\
\end{array}
\]

In particular, this is a deformation through Fredholm operators, and since index is locally constant it follows that \( \text{ind}(D) = \text{ind}(D_0) \). But \( D_0 \) is invertible, so the index is 0.

**Case 2:** We assume that the spectrum of \( L_0 + h_t \) is simple (each eigenspace has dimension one) and \( h_t \) are symmetric. Let \( \lambda_1(t), \ldots, \lambda_n(t) \) be the eigenvalues which ever cross 0, with eigenvectors \( u_1(t), \ldots, u_n(t) \) (these eigenvectors are functions of \( y \), but we’ll ignore that to keep notation relatively simple). If we pick an appropriate basis, we can write

\[
D = \frac{d}{dt} + \left( \begin{array}{c|c|c}
A(t) & \lambda(t) \\
\hline
\lambda(t) & \end{array} \right),
\]

where

\[
\lambda(t) = \begin{pmatrix}
\lambda_1(t) \\
\vdots \\
\lambda_n(t)
\end{pmatrix}
\]

and \( A \) is some infinite-dimensional diagonal matrix.

We now compute \( \ker(D) \). Let \( f \in \ker(D) \), and \( f = c_1 u_1 + \ldots + c_n u_n \) for \( c_1, \ldots, c_n \) functions of \( t \). The argument below can be used to show that any \( f \in \ker(D) \) has to be of this form as well.
We have

\[ 0 = Df(t) \]

\[ = \sum_{k=1}^{n} \frac{d}{dt} (c_k(t)u_k(t)) + (L_0 + h_t)(c_k(t)u_k(t)) \]

\[ = \sum_{k=1}^{n} \frac{dc_k}{dt}(t)u_k(t) + c(t)\frac{du}{dt}(t) + c_k(t)\lambda_k(t)u_k(t). \]

But \( \lambda_i \) and \( u_i \) are constant at infinity (by hypothesis, \( h_t \) is constant at infinity), so we can write

\[ \sum_{k=1}^{n} \left( \frac{dc_k}{dt}(t) + c_k(t)\lambda_k(\pm\infty) \right) u_k(\pm\infty) = 0 \]

for large enough \( t \). Since \( u_k(\pm\infty) \) are eigenvectors for \( h_\pm \), they are linearly independent, which implies

\[ \frac{dc}{dt}(t) + c(t)\lambda_k(\pm\infty) = 0 \]

for sufficiently large \( t \). At \(-\infty\), we get \( c_k(t) = d_k^- e^{-\lambda_k(-\infty)t} \), and at \(+\infty\) we have \( c_k(t) = d_k^+ e^{-\lambda_k(+\infty)t} \). In order for \( f \) to be \( L^2 \), we need \( \text{Re}(\lambda_k(-\infty)) < 0 \) and \( \text{Re}(\lambda_k(+\infty)) > 0 \). Thus the dimension of \( \ker(D) \) is the number of eigenvalues whose real part goes from negative to positive, which is precisely the number of intersections which count positively toward spectral flow.

A similar analysis will show that the dimension of \( \text{coker}(D) \) is the number of intersections which count negatively toward spectral flow. The basic idea is that we can define some kind of adjoint:

\[ D^* = -\frac{d}{dt} + L_0 + h_t^*. \]

Then the dimension of \( \text{coker}(D) \) is the dimension of \( \ker(D^*) \), so we only need to relate \( \ker(D^*) \) to the spectral flow. The relationship of elements of \( \ker(D^*) \) to changes in signs of eigenvectors is nearly identical to the argument above, except that some signs are switched. Thus we add to the dimension of \( \ker(D^*) \) when eigenvalues cross negatively. This completes the proof of case 2.

**Remark 1.** To see that the spectral index is finite, we need to note that the imaginary parts of the eigenvalues are bounded by the \( L^2 \) operator norm of \( h - h^* \). We know that the norms of \( h_t - h_t^* \) are uniformly bounded by continuity and compactness. Hence the eigenvalues in the spectral flow are contained in a band, finite in the imaginary direction. It then follows from the spectral theorem and basic principles of generic deformation that we can deform \( D \) to something which fits into case 2, up to the assumption that \( h_t \) is symmetric, which is probably not necessary for the argument.
We now proceed to a case which is of special interest in symplectic geometry. In what follows, we use \((s, t)\) coordinates in \(\mathbb{R} \times S^1\). We consider operators \(D : L^2_1(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \rightarrow L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})\) of the form

\[
D = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S,
\]

where \(J_0\) is the standard complex structure on \(\mathbb{R}^{2n}\) and \(S = S(s, t)\) is a smooth family of symmetric matrices satisfying

\[
\frac{\partial S}{\partial s}(s, t) = 0
\]

for \(|s| \gg 1\). We will see later that these operators arise as linearizations of the Floer equation. We endow \(\mathbb{R}^{2n}\) with the standard symplectic form \(\omega_{st} = \langle \cdot, J_0 \cdot \rangle\). Let \(Sp(2n)\) denote the Lie group of symplectic matrices, allow with a map \(sp(2n) \sim \rightarrow Symm, A \mapsto J_0 A\). By integrating \(S(s, t)\) in the \(t\)-direction we obtain a map \(\Psi : \mathbb{R} \times \mathbb{R} \rightarrow Sp(2n)\) such that

\[
\frac{d\Psi}{dt} = J_0 S \Psi \quad \text{and} \quad \Psi(s, 0) = I.
\]

Then there is some \(T\) such that \(\Psi(-s, t) = \Psi(-T, t)\) and \(\Psi(s, t) = \Psi(T, t)\) for all \(s > T\).

\[
\begin{array}{ccc}
\Psi(-T, t) & \Psi(s, t) & \Psi(T, t) \\
\Psi(s, 1) & \\
\Psi(s, 0) = I
\end{array}
\]

Let \(\gamma : [0, 1] \rightarrow Sp(2n)\) be a path such that \(\gamma(0) = I\) and 1 is not an eigenvalue of \(\gamma(1)\). Then we can define a Maslov (or Conley-Zehnder) index \(\mu_{CZ}(\gamma) \in \mathbb{Z}\). There are two ways to define it:

1. Let \(Sp^*(2n)\) be the set of symplectic matrices for which 1 is not an eigenvalue. The Maslov cycle \(Sp(2n) \setminus Sp^*(2n)\) is a singular codimension one subset, with singularities in codimension two.

The Maslov cycle is co-orientable, which means that any transverse crossing of it can be given a sign. Then we can define \(\mu_{CZ}\) to be the signed count of intersections with the Maslov cycle, for generic \(\gamma\).
(2) $Sp^*(2n)$ has two connected components; the component in which $M$ lies is determined by the sign of $\det(I - M)$. We identify a special point in each of these, $B_+ = -I$ and

$$B_- = \begin{pmatrix} 2 & \frac{1}{2} & & \\ \frac{1}{2} & -1 & & \\ & \ddots & \ddots & \\ & & & -1 \end{pmatrix}.$$ 

Any path $\gamma$ with $\gamma(1) \in Sp^*(2n)$ can be completed to a path $\tilde{\gamma}$ such that $\tilde{\gamma} \setminus \gamma$ doesn’t cross the Maslov cycle and $\tilde{\gamma}(1) = B_{\pm}$.

We know $U(n) = Sp(2n) \cap O(2n)$, and $U(n)$ is a maximal compact subgroup of $Sp(2n)$. Inclusion is a homotopy equivalence. If $\rho : Sp(2n) \to U(n)$ is a homotopy inverse to inclusion (in particular we use the one given by the polar decomposition of matrices in $Sp(2n)$), then $\det^2 \circ \rho \circ \tilde{\gamma}$ is a loop in $S^1$. Note that the $\det$ here is the complex determinant of matrices in $U(n)$, its absolute value squared gives the real determinant.

If we fix an isomorphism $\pi_1(S^1, 1) \cong \mathbb{Z}$, we can define $\mu_{CZ}(\gamma) = [\det^2 \circ \rho \circ \tilde{\gamma}]$.

The determinant map yields an isomorphism $\pi_1(Sp(2n), I) \to \pi_1(S^1, 1)$. If $\gamma(1) = I$, we can define $\mu_{CZ}(\gamma) = 2[\gamma]_{\pi_1}$. In particular, if $\gamma$ is contractible then $\mu_{CZ}(\gamma) = 0$. 

4