Caps and CZ index. From now on let’s work with contractible loops $L^0M$. Recall we had an index formula

$$\text{virtual dimension } M(x,y) = \mu(\Psi^-) - \mu(\Psi^+),$$

each term depends on a choice of (hermitian) trivialization of $u^*TM$. We remind the reader that $\Psi^\pm$ is the path of symplectic matrices induced by the Hamiltonian flow around the loop $x^\pm$.

The virtual dimension is the index of the linearization of the Floer equation:

$$\xi \mapsto \frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + S \xi.$$

One defect of the formula for the virtual dimension is that it requires you to trivialize $u^*TM$. The main point here is that a Hermitian bundle over $S^1$ can be trivialized but not canonically.

Two different trivializations of $u^*TM$ differ by multiplication by a map $\mathbb{R} \times S^1 \to U(n)$; Since $\pi_1(U(n)) \neq 0$, the Maslov class $\mu(\Psi^\pm)$ depends on the choice of trivialization.

On the other hand, trivializations over a disk of Hermitian bundles are homotopically unique (since $U(n)$ is connected).

Definition 1. A cap for a Hamiltonian trajectory $\gamma \in L^0M$ is a map $D^2 \to M$ extending $\gamma$. The importance of this is that we get

$$((\gamma, \text{cap}) \mapsto \text{well defined Maslov index } \mu(\gamma)).$$

It is defined by trivializing the pullback of $TM$ over $D^2$, and then considering the path of symplectic matrices induced by the Hamiltonian flow (with respect to this trivialization).

Claim 2. Let $\gamma$ be a Hamiltonian trajectory. If cap and cap’ are two caps of $\gamma$, then

$$\mu(\gamma, \text{cap}) - \mu(\gamma, \text{cap'}) = 2c_1(\text{cap} \cup \overline{\text{cap'}}),$$

where $c_1$ is the Chern class of $TM$ and the right-hand-side is the pairing of $c_1$ with the sphere cap $\cup \overline{\text{cap'}}$.

To see this use the interpretation of $c_1$ as an obstruction to extending sections of a line bundle. In our case the line bundle is $\text{det}_\mathbb{C}(TM)$.

Note that

$$\pi_0(\text{trivializations of } \gamma^*TM \text{ over } S^1) \simeq \pi_0(\text{trivializations of } \gamma^* \text{det}_\mathbb{C} TM).$$

There is a natural map $\rightarrow$, which we show is a bijection (using our knowledge of the fundamental group of $U(n)$).
Remark. Let \( E \to S^2 \) be a Hermitian bundle, trivialized over the two caps. The transition between the trivializations between cap and cap' is a function \( g : S^1 \to U(n) \). The determinant of this function is a function \( \det g : S^1 \to S^1 \), and therefore has a winding number. This winding number is the Chern class of the bundle (and is also the number of zeros of a transverse section of \( \det E \)).

Recall that we had a similar problem with “action:” \( A(\gamma) \) was not well defined, but

\[
A(\gamma, \text{cap}) = \int_{\text{cap}} \omega + \int_{\gamma} H \, dt,
\]

was well-defined. This implies that

\[
A(\gamma, \text{cap}) - A(\gamma, \text{cap}') = [\omega](\text{cap} \cup \text{cap}').
\]

In the monotone case when \( \tau[\omega] = c_1(M) \), \( \tau > 0 \), then the “ambiguities” in defining an action and a Maslov index are somehow proportional, and we have the following:

**Claim 3.** The quantity

\[
\eta(\gamma) = \mu(\gamma, \text{cap}) - 2\tau A(\gamma, \text{cap}),
\]

is independent of the cap. \( \square \)

The upshot of this claim is that given \( u \in M(x^\pm, H, J) \), we have

\[
\text{virdim}(M_u(x^\pm)) = \mu(\Psi^-) - \mu(\Psi^+)
\]

\[
= \mu(\Psi^-) - \mu(\Psi^+)
\]

\[
= \mu(x^-, \text{cap}) - \mu(x^+, \text{cap} \cup u)
\]

\[
= \eta(x^-) - \eta(x^+) - 2\tau(A(x^+, \text{cap} \cup u) - A(x^-, \text{cap}))
\]

\[
= \eta(x^-) - \eta(x^+) - 2\tau E(u).
\]

Here we are using the fact that

\[
E(u) = \frac{1}{2} \int |\frac{\partial u}{\partial x}|^2 = \int \omega + \int_{x^+} H \, dt - \int_{x^-} H \, dt.
\]

The geometrically defined energy is equal to a “topological” quantity.

**Remark.** This formula for the virtual dimension will be used to give an a priori energy bound on solutions \( u \) with virtual dimension 0 or 1.

**Theorem 4** (1.24). \( M(x^\pm, J, H) \) is a manifold (whose dimension is given by the virtual dimension), for generic \( J, H \).

**Remark.** As usual, we need to show that \( M \) is cut out by a section, and this section can be made transverse by generic choice of \( J, H \). Elliptic regularity will guarantee that the zero set of the section consists only of smooth maps \( u : \mathbb{R} \times S^1 \to M \).
Elliptic estimates. Let $\Omega' \subset \Omega \subset \mathbb{R}^n$ be bounded domains. Let $D$ elliptic differential operator of order $m$, then
\[ \|u\|_{W^{k+m, p}(\Omega')} \leq \text{const}(\| Du \|_{W^{k, p}(\Omega)} + \| u \|_{L^p(\Omega)}). \]
Note that when $Du = 0$, we get the elliptic bootstrapping procedure.

Remark. To use the elliptic estimates in a non-linear setting we need to work in coordinates. More precisely, we can try to find coordinates where the non-linearities become linear (for instance, finding coordinates which straighten out a vector field).

As another example, if $J$ is integrable, then we can find coordinates where the $J$-holomorphic equation becomes “linear.”

Remark. Another important thing to mention is that smooth solutions of the Floer equation are “automatically” in any Sobolev space of trajectories we might set-up, because of the exponential convergence at the ends.

Transversality. Fix $H$ and find $J$. We will have a Banach manifold of (compatible) complex structures $\mathcal{J}$, and a space of paths $\mathcal{P}$, and a parametrized family of sections
\[ \mathcal{J} \times \mathcal{P} \to TP \]
\[ (J, u) \mapsto \partial_s u + J(\partial_t u + X_H). \]
The linearization takes the form:
\[ (\xi, Y) \mapsto D_1(u, J)\xi + D_2(y, J)Y, \]
where $D_1(u, J)$ is the linearization of the Floer equation at $(u, J)$, while $D_2(u, J)$ will be just
\[ D_2(u, J) = YJ \frac{\partial u}{\partial s}. \]
We want to show that the linearization $(\ast)$ is surjective. To prove it is surjective, we argue by contradiction – if it weren’t surjective, we could find $\eta$ so that
\[ \int_{S^1 \times \mathbb{R}} \eta \cdot (D_1(u, J)\xi + D_2(y, J)Y) \, dsdt = 0, \quad \text{for all } \xi, Y. \]
The fact that $\eta \perp D_1(u, J)\xi$, for all $\xi$, implies that $\eta$ satisfies some sort of differential equation. In fact, it can be shown that $\eta$ satisfies the assumptions of Aronsajn’s lemma:

**Theorem 5** (Aronsajn). Let $\Omega \subset \mathbb{C}$ be a connected open domain. Suppose that $w \in W^{2, 2}_{\text{loc}}(\Omega, \mathbb{R}^{2n})$ satisfies
\[ |\Delta w| \leq C(|w| + |\partial_s w| + |\partial_t w|), \quad \text{almost everywhere} \]
Then if $w$ vanishes to all orders (?) at $z_0 \in \Omega$, then $w$ vanishes identically on $\Omega$.

Remark. This replaces the ODE “unique” continuation statement we used in our exploration of Morse theory. Therefore to prove that $\eta = 0$ we need only to prove that $\eta$ vanishes in the neighborhood of some point.
Now by setting $\xi = 0$, we will have an equation of the form

\[(\star) \quad \int \langle \eta, YJ \partial_s u \rangle = 0,\]

and so since $\frac{\partial u}{\partial s} \neq 0$ in some neighborhood, and since $\eta$ will also satisfy some ODE, we can sometimes use (\star) to prove that $\eta \equiv 0$.

We will focus on what doesn’t work. Suppose we have $(M^{2n}, \omega, J)$. Then we know that

\[
\text{virdim}(\text{spheres in } A \in H_2(M)) = 2n + 2c_1(A).
\]

If we don’t assume anything, it is possible that we have $J$ holomorphic spheres in a class $A$, with $c_1(A) < 0$. We will also therefore see holomorphic maps of the form

\[
S^2 \xrightarrow{z \mapsto z^k} S^2 \longrightarrow M,
\]

This demonstrates that using domain independent $J$ we cannot achieve transversality, because we can make the virtual dimension arbitrarily negative by taking multiple covers.

If we restrict to “somewhere-injective” curves then we can achieve transversality. The problem is multiply covered curves.

One solution is to use domain dependent $J$’s (where this multiple cover issue no longer works).

**Remark.** The problem in Floer theory is not in transversality, it’s in compactness. In general, there are “bubbled cylinders” appearing in the compactification of the moduli space.

This bubble is associated to a single point in the domain; as a consequence, the bubble is holomorphic for a domain independent $J$. These degenerations would break $d^2 = 0$.

Luckily, in the monotone case, we can eliminate these bubbles. In general, they seem inevitable.

As we will see a bit later, we can eliminate such things in a generic case. More precisely, if virdim = dim for all $J_{t,s}$-holomorphic spheres, then generically we would not have such breakings.

But we know (because of the multiple cover problem) that we cannot achieve this.

The solution to this problem (in general) is to count “bubbled” cylinders.