Examples of Bubbling:

For \([\alpha : \beta] \in \mathbb{CP}^1 \setminus [0 : 1]\), define map

\[
 u_{[\alpha : \beta]} : \mathbb{CP}^1 \to \mathbb{CP}^2 \\
 [x : y] \mapsto [\alpha x^2 : \alpha y^2 : \beta xy]
\]

As \(\beta \to 0\), \(u_{[\alpha : \beta]}\) converges in \(C^\infty\) topology to a double cover of \(\{z = 0\}\). As \(\alpha \to 0\), you can see bubbling. (After reparameterisation.)

First, let’s use the given parameterisation, then \(u_{[\alpha : \beta]}\) converges pointwisely to a constant map \([0 : 0 : 1]\), away from the points \([0 : 1]\) and \([1 : 0]\). In fact, \(u_{[\alpha : \beta]}|_{\mathbb{C}^*}\) converges in \(C^\infty_{loc}\) topology to the constant map.

Let’s compute \(|du_{[\alpha : \beta]}|\) at \([0 : 1]\).

Use \(\frac{x}{y}\) coordinates in \(\mathbb{CP}^1 \setminus [1 : 0]\), then \(\alpha y^2 \neq 0\) if \(\alpha \neq 0\), and use that affine chart for \(\mathbb{CP}^2\), then we get the following map, assuming \(\alpha \neq 0\):

\[
 u_{[\alpha : \beta]} : \mathbb{A}^1 \to \mathbb{A}^2 \\
 x \quad \mapsto \quad \left(\frac{x^2}{y^2}, \frac{\beta x}{\alpha y}\right)
\]
As $\alpha$ tends to $0$, $|du_{[\alpha : \beta]}|$ tends to $\infty$. (Similar happens at $[1 : 0]$)

To converge to a bubble, we need to reparameterise to squeeze into a neighbourhood of the point where $|du_{[\alpha : \beta]}|$ is large. Define $\tilde{u}_{[\alpha : \beta]}$ to be the composition of a reparameterisation $[x : y] \mapsto [\alpha x : \beta y]$ and $u_{[\alpha : \beta]}$. Notice, what this reparameterisation does is expand the hemisphere of $\mathbb{CP}^1$ containing the point $[1 : 0]$, and squeeze the other hemisphere, i.e., larger and larger portion of the sphere will be sent to a fixed small neighborhood of $[0 : 1]$, as $\alpha \to 0$.

$$\tilde{u}_{[\alpha : \beta]} : \mathbb{CP}^1 \to \mathbb{CP}^1 \to \mathbb{CP}^2$$

$$[x : y] \mapsto [\alpha x : \beta y] \mapsto [\alpha^3 x^2 : \alpha \beta y^2 : \alpha \beta^2 xy]$$

As $\alpha \to 0$, this map $\tilde{u}_{[\alpha : \beta]}$ converges in $C^\infty_{loc}$ topology to the map $[x : y] \mapsto [0 : x : y]$, away from $[1 : 0]$.

**Aside on a more nature definition of $J$-holomorphic curve equation and energy**

Suppose $(\Sigma, j)$ is a Riemann surface, and $(M, J)$ is an almost complex manifold. A $J$-holomorphic curve is a map $u : \Sigma \to M$, such that $du \circ j = J \circ du$.

Define the antiholomorphic part of the differential to be

$$\bar{\partial}_{j,J} u = \frac{1}{2} (du + J \circ du \circ j) \in \Omega^1(\Sigma, u^*TM),$$

and being holomorphic says $\bar{\partial}_{j,J} u = 0$.

In order to define energy of $u : \Sigma \to M$, we normally would need to choose a metric on $\sigma$ and $M$ respectively, and the energy would be

$$E(u) = \int_\sigma \text{trace}(du^*du)dvol$$

(*)
, where \( du^* \) is the adjoint of \( du \), and this is usually called the Dirichlet Energy. The cool thing is that this quantity doesn’t change if we change the metric \( g_\Sigma \) on \( \Sigma \) to \( e^f g_\Sigma \), when \( \Sigma \) is 2-dimensional, \textit{i.e.} it is a conformal invariance.

Which is even better when \( \Sigma \) is 2-dimensional, the choice of an orientation and a conformal structure is equivalent to the choice of a holomorphic structure. (Obviously, a holomorphic structure gives an orientation and a conformal structure. Conversely, an orientation and a conformal structure tell me how to rotate 90 degree counterclockwisely on each tangent space, which gives an almost complex structure on surface, which is an holomorphic structure in dimension 2.)

What is the integrand in \( \int \) in terms of \( j \)?

\[
E(u) = \int_\Sigma |\bar{\partial}_j J u|^2 \, d\text{vol} + \int_\sigma u^* \omega,
\]

if \( J \) is compatible with \( \omega \), a symplectic structure on \( M \), and \( g_M \) is the corresponding metric.

It has the following implications:

1. If \( u \) is \( J \)-holomorphic, the first term vanished, then it minimises the energy in the homology class, and it becomes a topology invariance, which only depends on the homology class \([u]\).

2. Energy is invariant under holomorphic reparameterisation.

**Floer equation and \( J \)-holomorphic curve equation**

Floer equation can be turned into \( J \)-holomorphic curve equation via the following Gromov’s trick.

Goal: Given \( X_H, J \) on \( M \), want to define an almost complex structure \( J_H \) on \( M \times T^*S^1 \).

\( pr : M \times T^*S^1 \longrightarrow T^*S^1 \) is a trivial bundle with symplectic fiber \( M \).

How to get a connection? Choose a 2-form \( \Omega \) on the total space, such that it restricts to the fiber direction is symplectic. Define

\[
L : T(M \times T^*S^1) \longrightarrow T^*M
\]

\[
v \longmapsto \Omega(v, -)
\]

Then, we define a subbundle \( H \) of \( T(M \times T^*S^1) \) by \( H = \ker L \), which satisfies \( TM \bigoplus H = T(M \times T^*S^1) \), as \( \Omega \) is symplectic when restricted to the vertical
tangent space. This gives an Ehresmann connection.

\[
\begin{array}{c}
M & \xrightarrow{\text{pr}} & M \times T^*S^1 \\
\downarrow \pi & & \downarrow \pi \\
T^*S^1 & & T^*S^1
\end{array}
\]

In our case, take \( \Omega = \text{pr}^*\omega + dH \wedge dt \). This gives an Ehresmann connection as described above, \( T(M \times T^*S^1) = TM \bigoplus H \), where \( H \) is isomorphic to \( T(T^*S^1) \), via the following horizontal lifting.

\[
\frac{\partial}{\partial s} \in T(T^*S^1) \mapsto \frac{\partial}{\partial s} \in T(M \times T^*S^1)
\]

\[
\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + X_H
\]

Then, we define our almost complex structure \( J_H \) to be the direct sum of the almost complex structure \( J \) on \( M \) and the almost complex structure \( j \) on \( T^*S^1 \), via the isomorphism \( T(M \times T^*S^1) \cong TM \bigoplus T(T^*S^1) \). In particular, \( \pi \) is a holomorphic map. Then there is a 1-1 correspondence between

\[ \{ \text{Holomorphic sections of } \pi \} \iff \{ \text{Solutions to Floer equation} \} \]

, which could be seen as follows.

A holomorphic section of \( \pi \) is a map \( U = (u, \text{id}) : T^*S^1 \rightarrow M \times T^*S^1 \), such that \( dU \circ j = J_H \circ dU \). Plug in \( \frac{\partial}{\partial t} \), we get

\[
J_H \circ dU \left( \frac{\partial}{\partial t} \right) = J_H \left( \frac{\partial u}{\partial t} + \frac{\partial}{\partial t} \right)
\]

\[= J_H \left( \frac{\partial u}{\partial t} - X_H \right) + \left( \frac{\partial}{\partial t} + X_H \right) \] (decompose into the form \( T(M \times T^*S^1) = TM \bigoplus H \))

\[= J \left( \frac{\partial u}{\partial t} - X_H \right) + j \frac{\partial}{\partial t} \] (using the definition of \( J_H \) and the isomorphism \( H \cong T(T^*S^1) \))

\[= J \left( \frac{\partial u}{\partial t} - X_H \right) - \frac{\partial}{\partial s} \] (using the isomorphism again to convert back to the space \( T(M \times T^*S^1) \)),

and \( dU \circ j \frac{\partial}{\partial t} = dU \left( - \frac{\partial}{\partial s} \right) = - \frac{\partial u}{\partial s} - \frac{\partial}{\partial s} \). So \( \frac{\partial u}{\partial s} = J \left( \frac{\partial u}{\partial t} - X_H \right) \), which is the Floer equation.

Conversely, suppose we have a solution \( u \) to the Floer equation, then \( J_H \circ dU \circ j \left( \frac{\partial}{\partial t} \right) = dU \circ j \left( \frac{\partial}{\partial t} \right) \), by the above calculation. As \( j \frac{\partial}{\partial t} = \frac{\partial}{\partial s} \), this holds for
For us, the upshot is that local properties of $J$-holomorphic curves can be translated to local properties of Floer solutions, e.g.:

Theorem (Removal of singularity): Let $D$ be a closed disk. Assume that $u : D \setminus \{0\} \to M$ is a $J$-holomorphic curve with finite energy, then there exists a unique extension $\tilde{u} : D \to M$ which is $J$-holomorphic.