Math 113: Midterm

April 30, 2021

Note: Unless specified in the question, all of your answers should be justified by complete and rigorous proofs.

Q1
i) Let \( \mathbb{F}_3 = \{0, 1, a\} \) be a field with three elements where 0 is the additive identity and 1 is the multiplicative identity. Fill out the following tables. (You do not need to explain your reasoning.)

\[
\begin{array}{c|ccc}
+ & 0 & 1 & a \\
0 & 0 & 1 & a \\
1 & 1 & 0 & \text{a} \\
a & a & \text{a} & 0 \\
\end{array}
\quad
\begin{array}{c|ccc}
\cdot & 0 & 1 & a \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & a \\
a & 0 & a & 1 \\
\text{a} & 0 & a & 1 \\
\end{array}
\]

ii) Prove that \( \mathbb{Q} \) doesn’t have a subfield with finitely many elements.

iii) What are the additive and multiplicative inverses of \( 1 + 5i \) in \( \mathbb{C} \)?

Q2
Let \( \mathbb{F} \) be a field, and consider the vector space \( \mathbb{F}^\infty = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{F}, i = 1, 2, \ldots\} \) with component-wise addition and scalar multiplication as usual.

Let \( V \subset \mathbb{F}^\infty \) be the subset of vectors \( (a_1, a_2, \ldots) \) such that \( a_{2j-1} = a_{2j} \) for every \( j = 1, 2, \ldots \).

i) Prove that \( V \) is a subspace of \( \mathbb{F}^\infty \).

ii) Is \( V \) finite dimensional?

Q3
Let \( V \) be an 8-dimensional vector space over an arbitrary field \( \mathbb{F} \). Let \( W \) and \( U \) be subspaces of \( V \) which have of dimensions 4 and 5, respectively.

i) What are the possible values of \( \dim(W + U) \)?

ii) Can \( W + U \) be a direct sum?

iii) Does there exist a surjective linear map \( T : V \to W \) with \( \text{null}(T) = U \)? What about \( \text{null}(T) = W \)?

Q4 True or False Please write True or False next to the statements below - no justification is required.

i) Let \( S_1 \) and \( S_2 \) be sets. A map \( f : S_1 \to S_2 \) is bijective if there exists a surjective map \( g : S_2 \to S_1 \) such that \( f \circ g = \text{id}_{S_2} \).
ii) Let $V$ and $W$ be vector spaces over $\mathbb{F}_2$ with $\dim(V) = 3$ and $\dim(W) = 2$. Then $\mathcal{L}(V, W)$ has 32 elements.

iii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. If there exists exactly one vector that maps to $(1,1,1)$ then $T$ is surjective.

iv) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. If there exists at most one vector that maps to $(1,1,1)$, then $T$ is injective.

v) All infinite dimensional vector spaces over the field $\mathbb{F}_2$ are isomorphic to each other.