Problem 4. Let $V_1, \ldots, V_k$ be smooth vector fields on $M$ such that $[V_i, V_j] = 0$ for every $i, j$. Let $S \subset M$ be a submanifold, and define

$$S_t := \varphi_{t_k}^k \cdots \varphi_{t_1}^1(S)$$

where $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$ and $\varphi_i$ is the flow of $V_i$.

Find a necessary and sufficient condition on $V_1, \ldots, V_k$ such that for any countable collection of manifolds $\{Y_j : j \in \mathbb{N}\}$ and smooth maps $f_j : Y_j \to M$, there exists a $t \in \mathbb{R}^k$ such that $S_t$ is transverse to $f_j$ for every $j$.

Solution. Our proof will use the following theorem:

Theorem 1 (parametric transversality). Let $f : Y \to M$ be a smooth map, and let $F : G \times \Sigma \to M$ be a smooth map which is transverse to $f$, i.e. for all $(y, g, s)$ such that $p = f(y) = F(g, s)$, $\text{im } df_y + \text{im } dF_{g,s} = TM_p$. Then there is a countable intersection of dense open sets, $G^{\text{reg}} \subset G$, so that

$$g \in G^{\text{reg}} \implies F|_{g \times \Sigma} : \Sigma \to M \text{ is transverse to } f : Y \to M.$$ 

Remark 2. A set which contains a countable intersection of dense open sets is called Baire generic. Clearly a countable intersection of Baire generic sets is still Baire generic. Baire’s theorem shows that Baire generic sets are dense.

This can be compared with the notion of Lebesgue generic – one says a set is Lebesgue generic if its complement is measure zero. There are examples in $[0, 1]$ of Lebesgue generic sets whose complements are also Baire generic, so these two notions of genericity do not agree in general. One example is to consider a countable intersection of sets of the form $\bigcup_n (q_n - \epsilon_n, q_n + \epsilon_n)$ where $q_n$ is an enumeration of the rationals, and $\epsilon_n$ is a sequence which sums to a small number.

However, if a set is open and Lebesgue generic then it is dense. One can easily show that the set of regular values of a map $f : \overline{B(1)} \to N$ is open, and so by the “Lebesgue generic” version of Sard’s theorem, this set of regular values is dense. By covering any manifold $M$ by countably many compact balls, one can thus conclude that the set of regular values of any map $f : M \to N$ is a countable intersection of dense open sets. Thus we conclude the “Baire generic” version of Sard’s lemma from the “Lebesgue generic” version of Sard’s lemma.

The desired necessary and sufficient condition on the vector fields is that they span the normal bundle to $S$, i.e. for all $p \in S$

$$(*) \quad TM_p = \text{span } \{V_{1,p}, \ldots, V_{k,p}\} + TS_p.$$ 

Remark 3. To prove that this is necessary, we will need to assume that the vector fields commute. One can concoct examples with $\dim S = n - 2$ in an $n$-manifold $M$ and non-commuting
$V_1, V_2,$ where condition $(\ast)$ is not necessary to ensure the conclusion of the problem. A particularly simple example is given in $S^2$, where one lets $S$ be a point.

To prove that this condition is sufficient, it is not necessary that the vector fields commute, although knowing that $[V_i, V_j] = 0$ does simplify the proof that $(\ast)$ is sufficient.

Now we prove that $(\ast)$ is sufficient. Consider any map $f : Y \to M$, and define

$$F : \mathbb{R}^k \times Y \to M \text{ given by } F(t, s) = \varphi_{-t}^1 \cdots \varphi_{-t_k}^k(f(y)).$$

We claim that $F$ is transverse to $S$. To see this, we will use the fact that the vector fields commute (cf Remark 3). Then it is easy to see that for all $t, y$

$$\text{im } dF_{t,y} \text{ contains span } \{V_{1,p}, \ldots, V_{k,p}\} \text{ where } p = F(t,y).$$

Therefore, if $F(t,y) \in S$, the condition $(\ast)$ implies that $\text{im } dF_{t,y} + TS_p = TM_p$, as desired.

Therefore for a generic set of $t \in \mathbb{R}^k$ we conclude that $y \mapsto F(t,y)$ and $S$ are transverse. Applying the global diffeomorphism $x \mapsto \varphi_k^t \cdots \varphi_1^t$, we conclude that $S_t$ and $y \mapsto F(0,y) = f(y)$ are transverse. Therefore we conclude a generic set of $t$ for which $S_t$ and $f$ are transverse.

Since a countable intersection of generic sets is generic, we conclude a generic set of $t$ for which $S_t$ and every function in $\{f_j : Y_j \to M : j \in \mathbb{N}\}$ are transverse. Thus $(\ast)$ is sufficient.

To see that $(\ast)$ is necessary, suppose that it fails at some point $p \in S$. Consider the orbit $f : Y \to M$ of the flow passing through $p$.

To be a bit more precise, $Y = \mathbb{R}^k$ and $f(t) = \varphi_k^t \cdots \varphi_1^t(p)$. It is clear that $\text{im } df_0$ is the span of the vector fields $V_{1,p}, \ldots, V_{k,p}$.

Therefore, by construction, $f$ and $S$ are not transverse at $f(0) = p$. If $f$ and $S_t$ were transverse everywhere, then in particular they would be transverse at

$$f(t) = \varphi_k^t \cdots \varphi_1^t(p) \in S_t.$$ 

By flowing backwards by $\varphi_1^{-t_1} \cdots \varphi_k^{-t_k}$ we would conclude that

$$(\dagger) \quad g = \varphi_1^{-t_1} \cdots \varphi_k^{-t_k} \circ f \text{ and } \varphi_1^{-t_1} \cdots \varphi_k^{-t_k}(S_t) = S \text{ are transverse at } g(t) = p.$$ 

However, $g = f(\cdot - t)$, because the flows commute. Therefore $(\dagger)$ is equivalent to $f$ and $S$ being transverse at $f(0) = p$, which we have just said is not true. Therefore we have arrived at a contradiction, and thus $(\ast)$ is necessary.