Problem 4. Let $V_1, \dots, V_k$ be smooth vector fields on M such that $[V_i, V_j] = 0$ for every i, j. Let $S \subset M$ be a submanifold, and define

$$S_t := \varphi_k^{t_k} \cdots \varphi_1^{t_1}(S)$$

where $t = (t_1, \dots, t_k) \in \mathbb{R}^k$ and φ_i is the flow of V_i.

Find a necessary and sufficient condition on $V_1, \dots, V_k$ such that for any countable collection of manifolds $\{Y_j : j \in \mathbb{N}\}$ and smooth maps $f_j : Y_j \to M$, there exists a $t \in \mathbb{R}^k$ such that S_t is transverse to f_j for every j.

Solution. Our proof will use the following theorem:

Theorem 1 (parametric transversality)**.** Let $f : Y \to M$ be a smooth map, and let

$$F : G \times \Sigma \to M$$

be a smooth map which is transverse to f, i.e. for all (y, g, s) such that $p = f(y) = F(g, s)$, $\operatorname{im} \mathrm{d}f_y + \operatorname{im} \mathrm{d}F_{g,s} = TM_p$. Then there is a countable intersection of dense open sets, $G^{\mathrm{reg}} \subset G$, so that

$$g \in G^{\mathrm{reg}} \implies F|_{g \times \Sigma} : \Sigma \to M \text{ is transverse to } f : Y \to M.$$

□

Remark 2. A set which contains a countable intersection of dense open sets is called **Baire generic**. Clearly a countable intersection of Baire generic sets is still Baire generic. Baire's theorem shows that Baire generic sets are dense.

This can be compared with the notion of **Lebesgue generic** – one says a set is Lebesgue generic if its complement is measure zero. There are examples in $[0, 1]$ of Lebesgue generic sets whose complements are also Baire generic, so these two notions of genericity do not agree in general. One example is to consider a countable intersection of sets of the form $\bigcup_n (q_n - \epsilon_n, q_n + \epsilon_n)$ where q_n is an enumeration of the rationals, and ϵ_n is a sequence which sums to a small number.

However, if a set is open and Lebesgue generic then it is dense. One can easily show that the set of regular values of a map $f : \overline{B(1)} \to N$ is open, and so by the "Lebesgue generic" version of Sard's theorem, this set of regular values is dense. By covering any manifold M by countably many compact balls, one can thus conclude that the set of regular values of any map $f : M \to N$ is a countable intersection of dense open sets. Thus we conclude the "Baire generic" version of Sard's lemma from the "Lebesgue generic" version of Sard's lemma. □

The desired necessary and sufficient condition on the vector fields is that they span the normal bundle to S, i.e. for all $p \in S$

$$(*) \qquad TM_p = \operatorname{span}\{V_{1,p}, \dots, V_{k,p}\} + TS_p.$$

Remark 3. To prove that this is necessary, we will need to assume that the vector fields commute. One can concoct examples with $\dim S = n-2$ in an n-manifold M and non-commuting

V_1, V_2, where condition $(*)$ is not necessary to ensure the conclusion of the problem. A particularly simple example is given in S^2, where one lets S be a point.

To prove that this condition is sufficient, it is not necessary that the vector fields commute, although knowing that $[V_i, V_j] = 0$ does simplify the proof that $(*)$ is sufficient. ◻

Now we prove that $(*)$ is sufficient. Consider any map $f : Y \to M$, and define

$$F : \mathbb{R}^k \times Y \to M \text{ given by } F(t, s) = \varphi_1^{-t_1} \cdots \varphi_k^{-t_k}(f(y)).$$

We claim that F is transverse to S. To see this, we will use the fact that the vector fields commute (cf Remark 3). Then it is easy to see that for all t, y

$$\operatorname{im} \mathrm{d}F_{t,y} \text{ contains span}\{V_{1,p}, \ldots, V_{k,p}\} \text{ where } p = F(t, y).$$

Therefore, if $F(t, y) \in S$, the condition $(*)$ implies that $\operatorname{im} \mathrm{d}F_{t,y} + TS_p = TM_p$, as desired. Therefore for a generic set of $t \in \mathbb{R}^k$ we conclude that $y \mapsto F(t, y)$ and S are transverse. Applying the global diffeomorhism $x \mapsto \varphi_k^{t_k} \cdots \varphi_1^{t_1}$, we conclude that S_t and $y \mapsto F(0, y) = f(y)$ are transverse. Therefore we conclude a generic set of t for which S_t and f are transverse.

Since a countable intersection of generic sets is generic, we conclude a generic set of t for which S_t and every function in $\{f_j : Y_j \to M : j \in \mathbb{N}\}$ are transverse. Thus $(*)$ is sufficient.

To see that $(*)$ is necessary, suppose that it fails at some point $p \in S$. Consider the orbit $f : Y \to M$ of the flow passing through p.

To be a bit more precise, $Y = \mathbb{R}^k$ and $f(t) = \varphi_k^{t_k} \cdots \varphi_1^{t_1}(p)$. It is clear that $\operatorname{im} \mathrm{d}f_0$ is the span of the vector fields $V_{1,p}, \ldots, V_{k,p}$.

Therefore, by construction, f and S are not transverse at $f(0) = p$. If f and S_t were transverse everywhere, then in particular they would be transverse at

$$f(t) = \varphi_k^{t_k} \cdots \varphi_1^{t_1}(p) \in S_t.$$

By flowing backwards by $\varphi_1^{-t_1} \cdots \varphi_k^{-t_k}$ we would conclude that

$$(\dagger) \qquad g = \varphi_1^{-t_1} \cdots \varphi_k^{-t_k} \circ f \text{ and } \varphi_1^{-t_1} \cdots \varphi_k^{-t_k}(S_t) = S \text{ are transverse at } g(t) = p.$$

However, $g = f(\cdot - t)$, because the flows commute. Therefore $(\dagger)$ is equivalent to f and S being transverse at $f(0) = p$, which we have just said is not true. Therefore we have arrived at a contradiction, and thus $(*)$ is necessary. ◻