Problem Set 1

Math 257a

October 4, 2019

Let M be an n-dimensional closed smooth manifold. In this problem set, we call a degree n differential form on M a *volume form* if it is non-zero everywhere on M.

1. Find a necessary and sufficient condition on the top deRham cohomology $H^n_{dR}(M)$ for M to admit a volume form. Please don't just copy a solution, do it yourself after you refresh yourself on definitions.

2. Prove that the unit sphere S^n in $\mathbb{R}^{n+1}$ is a submanifold, and hence it has an induced smooth manifold structure. Write down an explicit volume form on S^n in coordinates using that smooth structure. More precisely, cover S^n by some explicit charts in its atlas, and write down an n-form in the coordinates of each of these charts which glue to a volume form on S^n.

3. For which values of n does S^n admit a symplectic form?

4. Let $V_1, \dots V_k$ be smooth vector fields on M, such that the Lie bracket $[V_i, V_j] = 0$ for every $i, j \in \{1, \dots, k\}$. Let $S \subset M$ be a submanifold, and define

$$S_t := \phi_k^{t_k}(\cdots \phi_2^{t_2}(\phi_1^{t_1}(S)) \dots),$$

   where $t = (t_1, \dots t_k) \in \mathbb{R}^k$ and ϕ_i is the flow of V_i.

   Find a necessary and sufficient condition on $V_1, \dots V_k$ such that for any countable collection of manifolds Y_j and smooth maps $f_j : Y_j \to M$, there exists a $t \in \mathbb{R}^k$, such that S_t is transverse to f_j for every j.

5. Sketch the proof of Frobenius integrability theorem, which gives a simple necessary and sufficient condition for a distribution on a smooth manifold to be the tangent space distribution of a foliation. Feel free to use any resource you want but write down the sketch of proof by yourself.