Problem Set 3
Math 215b
January 31, 2019

1. Assume that $V$ is a smooth manifold with a smooth vector space structure, i.e. scalar multiplication $\mathbb{R} \times V \to V$ and vector addition $V \times V \to V$ are smooth maps. Does there have to be a linear diffeomorphism $V \to \mathbb{R}^n$ for some $n$?

2. Let $E \to B$ a vector bundle. Prove that if $S \subset E$ is a submanifold such that the intersection with each fiber is a subspace, then $S \to B$ is a vector bundle.

3. Give transition map descriptions of $E \otimes E'$, $\text{Hom}(E, E')$, $\Lambda^n E$, $E/S$, where $E \to B$ (rank $n$) and $E' \to B$ are vector bundles, and $S$ is subbundle of $E$.

4. (a) Equip $Gr(k, n)$, the set of all $k$ dimensional subspaces of $\mathbb{R}^n$, with a natural smooth manifold structure.
   
   (b) Define the tautological bundle $\tau \to Gr(k, n)$ as
   
   $$\tau = \{(l, x) \mid l \in Gr(k, n), x \in l\} \subset Gr(k, n) \times \mathbb{R}^n,$$

   and $(l, x) \mapsto l$. Prove that this is a smooth vector bundle.

5. Let $M^k \subset \mathbb{R}^n$ be a submanifold. We can define a map $G : M \to Gr(k, n)$ by sending $m$ to $T_m M \subset \mathbb{R}^n$. What is $G^* \tau$?