Problem Set 3

Math 257a

October 19, 2019

1. Let M be a closed manifold with a pseudo-Riemannian metric g (if it helps you, you can assume it is Riemannian, but positive definiteness is not relevant here). Let $a, b \in M$. One can define the geodesic equation for paths $\gamma : [0,1] \to M$ with $\gamma(0) = a$ and $\gamma(1) = b$ as the solution of a variational problem (least action principle); more specifically, as the solutions of the Euler-Lagrange equation for the action functional
$$\int_{[0,1]} g(\gamma'(t), \gamma'(t))dt.$$
Derive this equation in coordinates (we do not really care about a and b anymore, they were only necessary for the passage to the Euler-Lagrange equations) and check that it is correct using some resource. You should arrive at an equation that involves the Christoffel symbols of the metric. Incidentally, what would happen if g was not assumed to be symmetric, and just non-degenerate, e.g. a symplectic form?

2. Interpret the geodesic equation as the integral curve equation of a vector field on TM using the standard trick to turn a second order equation to a first order one. The flow of this vector field is called the geodesic flow. Note that g defines an isomorphism of vector bundles $TM \to T^*M$, which we think of as a diffeomorphism. Using this diffeomorphism we can define the co-geodesic flow on T^*M. Prove that this is a Hamiltonian flow. Hint: You might first want to guess the Hamiltonian, rather than proving it directly. The quantity $g(\gamma'(t), \gamma'(t))$ along a geodesic γ is constant, what is constant in a Hamiltonian flow?

3. The story from the last two questions is in fact an instance of a more general equivalence between dynamics ("a point moving in a system") defined i) on the configuration space via the least action principle for a Lagrangian action of a special kind (ultimately using Euler-Lagrange equations) and ii) on the phase space via Hamilton's equations for a corresponding Hamiltonian. The link between the two is what is called the Legendre transformation. Read about this and explain the geodesic story in these terms. Section 7.5 of the book "Global Analysis: Differential Forms in Analysis, Geometry and Physics" seems to have all of this in a readable form, culminating in Theorem 21.

4. Find the thesis of Paul Seidel "Floer homology and the symplectic isotopy problem", which is available on his website. Explain what is going on in Lemma 2.1 a). Prove that the same construction works for cotangent bundles of spheres of any dimension. Make the connection with the Dehn twist on an annulus that we defined in class. Can you think of any other cotangent bundle where such a compactly supported diffeomorphism can be defined?

5. Find the article "Convex symplectic manifolds" by Yakov Eliashberg and Mikhael Gromov. Explain Examples (a) and (b) in Section 1.12 "Weinstein manifolds".