1 Textbook problems

We will not be releasing solutions for the textbook problems. If you would like to see a solution to a specific problem, feel free to ask during office hours.

2 Polynomial and functional vector spaces

Beware! In this solution the letter $k$ is used in place of $\mathbb{F}$.

We note that a potentially helpful way of thinking of $k^\mathbb{Z}$ is as the set of bi-infinite vectors $\{(..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...): a_i \in k, i \in \mathbb{Z}\}$, where the $i$-th position corresponds to the output of the function when evaluated at $i$.

i) Let $f_i$ be the function defined by

$$f_i(j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

(Think of $f_i$ as the analog of the $i$-th standard coordinate vector in the interpretation of $k^\mathbb{Z}$ presented above.) For every $n$, the set $\{f_1, f_2, ..., f_n\}$ is linearly independent. Indeed, suppose that $a_1f_1 + ... + a_nf_n = 0$ for some $a_i \in k$. (Note: This means that the LHS is the zero function, i.e. evaluates to 0 on any input.) By definition of $f_i$, for all $i$ we have

$$0 = (a_1f_1 + ... + a_nf_n)(i) = a_1f_1(i) + ... + a_nf_n(i) = a_1 \cdot 0 + ... + a_i \cdot 1 + ... + a_n \cdot 0 = a_i.$$ 

It follows that $a_i = 0$ for all $i$, so $f_1, ..., f_n$ are LI. By Problem 14 from the text, $k^\mathbb{Z}$ is infinite dimensional.

ii) Suppose that $P(k)$ is finite dimensional and let $\{p_1, ..., p_n\}$ be a spanning set. Let $d_i = \deg(p_i)$ and let $D = \max_{i=1,...,n} d_i$. Then for any $a_i \in k$, $\deg(a_1p_1 + ... + a_np_n) \leq D$. In particular, $x^{D+1} \notin \text{span}\{p_1, ..., p_n\}$. This contradicts our assumption that the $p_i$ form a spanning set, so $P(k)$ must be infinite dimensional.

Let $P_d(k) = \{a_0 + a_1x + ... + a_dx^d : a_i \in k\}$ be the subspace of polynomials of degree at most $d$. It can be shown that $P_d(k)$ and $P_{d'}(k)$ are not isomorphic when $d \neq d'$ using techniques similar to PSet 2. (In
particular, one can show that a linearly independent set in $P_d(k)$ has size at most $d + 1$. Furthermore, the set $X_d = \{1, x, \ldots, x^d\}$ is linearly independent. If there were an isomorphism $\phi: P_d(k) \rightarrow P_d(k)$ with $d > d'$, then the set $\phi(X_d) \subset P_d(k)$ would be a linearly independent subset of size $d + 1 > d' + 1$, a contradiction.

Finally, we show that it is impossible to find finitely many finite-dimensional subspaces of $P(k)$ whose sum is $P(k)$. Let $V_1, \ldots, V_n$ be finite dimensional subspaces of $P(k)$ where $V_i$ has spanning set $\{p_i^{(1)}, \ldots, p_i^{(k)}\}$. Then the set $\{p_j^{(i)}\}_{i,j}$ spans the sum $V_1 + \cdots + V_n$, and by the same degree argument used in the first part of this question, this cannot span the entirety of $P(k)$.

**Remark** (Not required for full credit on the question)

As an aside, in exactly what sense $x^i$ are linearly independent? Over an arbitrary field $k$, these may not actually be linearly independent as functions. For instance, in $\mathbb{F}_p$, $x^p$ and $x$ take on the same value at all points by Fermat’s little theorem. However, algebraically we still consider them to be linearly independent by simply considering at all points by Fermat’s little theorem. For instance, in $\mathbb{F}_p$, we cannot have $x^p = x$ for all $x$ in $\mathbb{F}_p$.

Consider any $k^S$ that is finite dimensional, then $S$ must be finite. We will prove the contrapositive, namely that if $S$ is infinite then $k^S$ is infinite dimensional. If $S$ is infinite, then it has a countable set of distinct elements $s_1, s_2, \ldots$. Let

$$f_i(s) = \begin{cases} 1 & s = s_i \\ 0 & s \neq s_i \end{cases}$$

By the same logic as in part i), the collection $\{f_1, \ldots, f_n\}$ is LI for all $n$, so again by problem 14 $k^S$ is infinite dimensional.

(Note: The converse of this statement is true, so $k^S$ is finite dimensional if and only $S$ is finite. See if you can prove this statement!)

iv) $P(k)$ and $k^\mathbb{N}$ are not isomorphic. It suffices to show that $k^\mathbb{N}$ cannot be spanned by a countable set of vectors. (Since $\{x^i : i \geq 0\}$ span $P(k)$, if we $\phi: P(k) \rightarrow k^\mathbb{N}$ were an isomorphism then $\{\phi(x^i) : i \geq 0\}$ would span $k^\mathbb{N}$, a contradiction.)

Consider $k^\mathbb{N}$ as a product of $k^{i+1}$ for $i \geq 1$ (group the first two coordinates together, the third after that together, four after that together etc.). Denote the projection from $k^\mathbb{N}$ to $k^{i+1}$ by $\pi_i$. Consider any countable set of vectors $y_1, y_2, \ldots \in k^\mathbb{N}$. We will construct an element in $k^\mathbb{N}$ that is not in their span by a diagonal trick. For any $i \geq 1$, we can find a vector in $k^{i+1}$ which is not in the span of $\pi_i(y_1), \ldots, \pi_i(y_i)$. (This is true since the dimension of $k^{i+1}$ is $i + 1$ and there are only $i$ vectors in this list.) Choose any vector not in the span and call this vector $z_i$. 

\[\begin{array}{c}\end{array}\]
We now claim that \( z = (z_1, z_2, z_3, \ldots) \) is not in the span of the \( y_i \). If it were, then it would be a linear combination of \( y_1, \ldots, y_j \) for some \( j \). Then in particular we would need to have that \( z_j \) is a linear combination of \( \pi_j(y_1), \ldots, \pi_j(y_j) \). By construction this is not the case.