Problem Set 4
Math 113: Linear Algebra and Matrix Theory
If you notice any mistakes, please email the CA: ddore@stanford.edu
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1 Textbook problems
2B: 4, 7, 8
2C: 3, 10, 14, 15

2 Matrix multiplication
Let \( A \) be an \( m \times n \) matrix and \( B \) be a \( k \times m \) matrix. Consider the linear maps \( T_A : \mathbb{F}^n \to \mathbb{F}^m \) and \( T_B : \mathbb{F}^m \to \mathbb{F}^k \) as defined in Lecture 11. Prove that
\[
T_B \circ T_A = T_{BA}
\]
as linear maps \( \mathbb{F}^n \to \mathbb{F}^k \), where we \( BA \) is the matrix multiplication of \( B \) and \( A \).

3 Finite fields revisited
Let \( \mathbb{F} \) be a field with finitely many elements. Recall the definition of \( c_\mathbb{F} \) from your first problem set before you proceed.

i) Prove that \( c_\mathbb{F} \) must be a prime number.

ii) Let \( c_\mathbb{F} = p \). Show that \( \mathbb{F} \) can be equipped with a scalar multiplication and vector addition which makes it a vector space over \( \mathbb{F}_p \). (Hint: You might try to find \( \mathbb{F}_p \) as a subfield inside \( \mathbb{F} \) in order to define the vector space structure, similar to how \( \mathbb{R} \) was a vector space over \( \mathbb{Q} \) in PSet 2.)

iii) Show that \( |\mathbb{F}| = p^n \) for some positive integer \( n \).

iv) Construct a field with 4 elements.

Remarks (You do not have to prove these statements, they are just remarks)

• There is exactly one field with \( p^n \) elements for each prime \( p \) and positive integer \( n \).
The fields where $1 + 1 + \cdots + 1 = 0$ for some $m > 1$ are called finite characteristic fields. They do not have to have finitely many elements. One can again show that the smallest such $m$ has to be a prime $p$. Therefore, if the field is not finite, we have an infinite dimensional vector space over $\mathbb{F}_p$.

If you have free time, you might want to think or read about fields with $p^n$ elements or infinite fields with finite characteristic.