1 Textbook problems

We will not be releasing solutions for the textbook problems. If you would like to see a solution to a specific problem, feel free to ask during office hours.

2 Determinants

i) First, observe that
\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{bmatrix}
=\begin{bmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
= (a_{11}a_{22} - a_{12}a_{21}) \begin{bmatrix}1 & 0 \\0 & 1\end{bmatrix}
\] (1)

Thus if \(a_{11}a_{22} - a_{12}a_{21} \neq 0\), we see that \(A\) is invertible with
\[
A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{bmatrix}
\]

Conversely, suppose \(a_{11}a_{22} - a_{12}a_{21} = 0\). We divide our problem into two cases. If \(a_{ij} = 0\) for all \(i, j\) then \(A\) is the zero matrix and is clearly not invertible. Otherwise, either \(v = \begin{bmatrix}a_{22} \\ -a_{21}\end{bmatrix} \neq 0\) or \(\begin{bmatrix}-a_{12} \\ a_{11}\end{bmatrix} \neq 0\);

WLOG, assume that the former statement is true. Then equation (1) tells us that
\(Av = 0\) with \(v \neq 0\). It follows that \(A\) is not injective and therefore not invertible.

ii) Wikipedia and Wolfram MathWorld, and Art of Problem Solving all have good resources on this.

iii) A direct calculation shows that
\[
\det(AB) = \det(A) \det(B)
\]

for any two \(2 \times 2\) matrices \(A\) and \(B\). In particular, since \(\det(A)\) and \(\det(B)\) are simply real numbers, their product commutes, so we have
\[
\det(AB) = \det(A) \det(B) = \det(B) \det(A) = \det(BA).
\] (2)
We have shown previously that there exists an invertible matrix $S$ such that $M'(T) = SM(T)S^{-1}$. (Change of basis formula.) Setting $A = SM(T)$ and $B = S^{-1}$, equation (2) tells us that

$$
det(M'(T)) = det((SM(T))S^{-1})
= det(S^{-1}(SM(T)))
= det((S^{-1}S)M(T))
= det(IM(T))
= det(M(T))
$$

as desired.