1 A weird vector space

We can make \( \mathbb{R} \) into a vector space over \( \mathbb{Q} \). The vector addition is the usual addition in \( \mathbb{R} \), and scalar multiplication is given by multiplying a real number with a rational number in the usual way.

i) Prove that this makes \( \mathbb{R} \) a vector space over \( \mathbb{Q} \).

ii) Prove that \( \mathbb{Q} \subset \mathbb{R} \) is a subspace of \( \mathbb{R} \) as a vector space over \( \mathbb{Q} \).

iii) Find a subspace \( V \) such that \( \mathbb{Q} \subset V \subset \mathbb{R} \). You can use without proof that \( \sqrt{2}, \sqrt{3}, \) and \( \sqrt{6} \) are not rational numbers.

iv) Can you find infinitely many subspaces \( V_1, V_2, \ldots \) of \( \mathbb{R} \) such that for every positive integer \( i \), \( V_i \subset V_{i+1} \)? You do not need to rigorously prove your statement but indicate why you think your answer is true. If you think the answer is yes, you should at least provide a candidate example. Feel free to read about transcendental numbers, and use that there exists transcendental numbers for this part.

2 Subspaces of \( \mathbb{F}_2^5 \)

i) Let \( V \subset \mathbb{F}_2^5 \) be a subspace. What are the possible values of \( |V| \)?

ii) Find subspaces \( V_1, V_2, V_3 \) of \( \mathbb{F}_2^5 \) such that \( V_1 \cap V_2 = V_2 \cap V_3 = V_1 \cap V_3 = \{0\} \) but \( V_1 + V_2 + V_3 \) is not a direct sum.

3 When do we consider two vector spaces to be the same?

Let \( \mathbb{F} \) be a field and let \( V, W \) be two vector spaces over \( \mathbb{F} \). An isomorphism from \( V \) to \( W \) is a bijective (meaning one-to-one and onto) map \( \phi : V \to W \) such that \( \phi(v + v') = \phi(v) + \phi(v') \) for all \( v, v' \in V \) and \( \phi(c \cdot v) = c \cdot \phi(v) \) for all \( c \in \mathbb{F} \) and all \( v \in V \).

i) Prove that if there is an isomorphism \( V \to W \) then there is also an isomorphism \( W \to V \). In this case we say that \( V \) and \( W \) are isomorphic.

In the following two parts, \( \mathbb{F} \) means the vector space \( \mathbb{F}^1 \).
ii) Prove that the subspace \( \{(t,t,t) \mid t \in \mathbb{F} \} \subset \mathbb{F}^3 \) and \( \mathbb{F} \) are isomorphic.

iii) Prove that \( \mathbb{F} \) and \( \mathbb{F}^2 \) are not isomorphic.

iv) Find subspaces \( U, U', V, V' \subset \mathbb{R}^3 \) such that no two of them are isomorphic as vector spaces, but \( U + U' \) and \( V + V' \) are isomorphic. You can use your intuition about lines and planes in three dimensional space as long as you understand their connection to our definitions in this problem. Example of the kind of fact you can use: if you have a point \( p \) on a line, then the line contains other points which are not equal to \( p \).