I) Nodal integral affine manifolds
   1) one node models vs. nodal cut-affine models
   2) nodal slide symplectomorphisms
   3) ziegenvay diagrams and Vienna mutations

II) Symplectic cluster manifolds
   1) relation to cluster varieties
   2) relation to toric models

III) Relative symplectic cohomology
   1) compact sets
   2) open sets
   3) locality
   4) $T^*T^n \cong M_0$ case
   5) $M_1$ and the canonical pair of embeddings $M_0 \hookrightarrow M_1$ (wall-crossing)
   6) The sheaf in the base of a symplectic cluster manifold.
   7) The computation of the sheaf on $B_1$. 
IV) Mirror construction
   1) cluster construction
   2) non-archimedean SYZ fibrations constructions.

IV) Homological mirror symmetry
      \{ future work \}
      1) open string theory
      2) construction of the functor.

      Joint work with Joel Gromov
Quick plan (some of these are not yet written)

- Introduce singularity diagrams and the corresponding symplectic manifolds
- Justify \( \sim \) calling these symplectic "cluster" manifolds
- Explain a locality theorem for relative symplectic cohomology
- Explain local computation about \( \overline{B}_0 \) and \( \overline{B}_1 \) (well crossing were)
- Construct an algebraic mirror which for \( a_i = 0 \) are the "cluster variety" related to \( \mathbb{A}_n \)
- Construct an analytic mirror that is the analytification of the algebraic one which also admits a nonarch 5-12 fibration
I. Nodal integral affine manifolds.

1) Let $B_k$, $k \in \mathbb{Z}$, denote the integral affine structure on a topological $\mathbb{R}^2$-$\mathbb{E}^2$ given by straight lines.

A nodal integral affine manifold is a topological manifold $B$ with a finite number of nodes $NCB$, and an integral affine structure on
\( B^m := B - N \) such that every \( \text{n e} N \) admits a punctured neighborhood integral affine iso to the punct. nghb. of origin in \( B^m \).

A final condition can be checked by computing affine monodromy though it will be automatic for \( u \).

\( B \) mod. integral affine manifold

\( \quad \Rightarrow \)

symplectic manifold

\( \quad ( = T^* B^m ) \cup \) local models

\( M_B \)

\( M_B \) also admits a Lagrangian fibration with focus-focus singularities above the nodes.
Let us also introduce the rotating $M_k$ for $\tilde{M}_k$.

2) Consider the following two models. Integral affine manifolds (with a naive identification as sets).

There exists a symplectomorphism $M_B \cong \tilde{M}_B$ which agrees with the naive identification outside of $\mathcal{U} \tilde{\mathcal{U}}$. 
This can be implanted in more global situations. We call them model slide symplectomorphisms.

3) An eigenray diagram is a disjoint union of rays with rational slope inside $\mathbb{R}^2 = (\mathbb{R} \otimes \mathbb{Z}^2)$

"Symington cuts"

The data also includes possibly more "nodes" along the eigenrays.
and an integer at each node.

An eigenvray diagram \( R \) has an explicit raying description:

\[
\begin{array}{c}
B_R \\
\rightarrow \\
M_R
\end{array}
\]

nodal int. eff refl. symplectic manifold.

For simplicity assume no more than one node on rays, that no two rays are parallel, and all numbers are positive.

\( \Delta \) Sliding the nodes along the eigenvrays do not change the symplectic manifold.

Def: We say that in an eigenvray diagram a node is in suitable position if at that node the
ray that extends to the other side of the eigenray does not intersect any of the other eigenrays.

Not in mutable position

Now in mutable position.
By sliding the other nodes we can make any node come to a mutable position. Then we can apply a branch move to get another energy ray diagram which describes the same symplectic manifold.

"Vianna mutation"

All describe the same symplectic manifold.
II - Symplectic cluster manifolds

1) Notice that an eigenray diagram gives rise to a seed data (with or simplifying assumption)

Consider $\mathbb{Z}^2$ skew symmetric bilinear form given by determinant

$$\omega((a, b), (c, d)) = ad - bc.$$ 

This only depends on the orientation of $\mathbb{Z}^2$. (can be pulled back to $\mathbb{Z}^2 \to \mathbb{Z}^2$)
The matrices

\[
\begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix}
\]

can be expressed in more invariant notation as the linear maps.

\[v \mapsto v + k\omega(u, v) n,\]

where \(n\) is the primitive vector along the eigenvector.

So the data of an eigenvector diagram is a bunch of vectors \((k, n)\) in \(\mathbb{Z}^2/\omega\) and some numbers \(\omega(n_i, v_i) \in \mathbb{R}\), (up to sliding the nodes).

The numbers \(\alpha_i\) measure the failure of the eigenvrays to pass
through the origin. This is related to the exactness of the symplectic form on $\mathcal{M}_R$. This seed data can be used to construct $\gamma_R$ and $\mathcal{A}_R$ cluster varieties with no frozen variables and "rank 2".

**Conjecture:**

\[ \gamma_R \]
\[ \begin{array}{c}
\uparrow \\
\mathcal{A}_R
\end{array} \]

The fibre above $e$ called $\mathcal{U}_R$ has holomorphic symplectic structure $\omega$. It seems plausible that $\mathcal{M}_R$ would be symplectomorphic to the "holes filled" version of $(\mathcal{U}_R, \text{im}(\omega))$.
By construction, this \( U \) has an almost cover by \((C^*)^2\) charts with
\[
(c_1^2) = \frac{b_{21}}{\bar{b}_{11}} \wedge \frac{b_{22}}{\bar{b}_{22}}.
\]

...glued to each other via rational maps of the form:
\[
(c_1^2, c_2^2) \mapsto (c_1^2, c_2^2(1 + c_1^2)^k)
\]

For us it is better to think that there is some variety \(Y_k\)

\[k > 1 \text{ not written earlier, remark later}
\]

...so embedded s.t. the induced rational map is the natural map as above.
If you consider $\text{Im}(\mathcal{S}) \times \text{Im}(\mathcal{R})$ in the $(\mathbb{C}^2)^2$ charts, you see that

$$(\mathbb{C}^2)^2, \text{Im}(\mathcal{S}) \big|_{\mathbb{C}^2} \sim (\mathbb{R}^2, \text{Im}(\mathcal{R})) \sim (\mathbb{R}^2, \text{distant}) \sim M_0.$$ 

Subconjecture: $(Y_k, \text{Im}(\mathcal{R}_k)) \sim M_k$

(true for $k=0,1$)

- $(Y_0, \text{Im}(\mathcal{R}_0)) \hookrightarrow (Y_k, \text{Im}(\mathcal{R}_k))$
- $(Y_0, \text{Im}(\mathcal{R}_0))$ is $\mathcal{C}$ up to from $\mathcal{C}$ is $M_0 \hookrightarrow M_k$ (defined later)

Here $(Y_k, \text{Im}(\mathcal{R}))$ can be thought of as being glued from $M_k$'s with intersections along $M_0$'s.
Better thought of as

We do not use this conjectural picture. It turns out one can construct
such a mutation tree directly on \( M_R \) by sliding the nodes to infinity and doing Vianna mutation. (I will come back to this.)

There are different ways to deform \((U, \text{Im}(\mathcal{N}))\)

1) look at other fillings of \( X \)

2) do symplectic reduction for other values in the moment map of \( T^* G A \) for element of \( C \)

3) \((z_1, z_2) \rightarrow (z_1, z_2 (1 + \lambda z_1)^k)\)

These must include R's with non-zero reals \( d_i \), but I do not know the precise statement yet.
Def: Any symplectic manifold symplectomorphic to $M_R$ for some origami diagram $R$ is called a symplectic cluster manifold.

We have a symplectomorphism

$$M = M_R$$

We automatically obtain one symplectic embeddy

$$z_R : M_0 \hookrightarrow M$$

& $|I|$ symplectic embeddys.

$$z_{R,i} : M_{k_i} \hookrightarrow M.$$ 

by sliding notes to $\infty$. 

($I$ is the set of notes)
(One can also use a certain Liouville flow argument here)
We can also keep some of the nodes fixed.

The images of $r_k, i_k$ give an open cover of $M_k$, any two of which intersect along the image of $r_k$. Let us call this the $R$-adapted cover of $M_k$.

(each induced embedding $M_0 \rightarrow M_k$ is a composition standard embedding translation by $r_k$)

Moreover, using Vianna's mutation, we can construct other derived diagrams $R'$ with $M_0 \rightarrow M_{k'}$.

And get $z_{R'} : M_0 \rightarrow M$

This constructs the promised mutation tree.
2) In literature the following procedure is used to obtain symplectic manifolds from cluster varieties.

1) Find a toric model for your cluster variety (non-toric blow-up's of toric varieties)

2) Find a Kähler form on the blow-up

3) Do "inflation"

I have never seen the details of this procedure in a meaningful generality.
We could instead use the eigenvector diagram approach.
- its scope is more general general than the approach above.
- it's entirely precise.
- can do Floer theory using monotonicity techniques (developed by Griman, Sikora, ...)

Example. Consider

\[
\text{inflation} \quad \left( \begin{array}{cc}
\frac{1}{x} & -
\end{array} \right)
\]
You just have to make a branch move.

Further comments:

- We are not interested in the divisor or the finite symplectic structures.
- Blow-up in symplectic geometry is very different from algebraic geo. You don’t really blow-up at a point, you remove a ball of a certain size and do “boundary reduction.”

The “parameter” is mostly the size of the ball, but it’s more...
complicated as max size ball at different points are different. How this parameter is carried from the "inflation" I have no idea.

Conclusion: rather than renormalizing triangles from a polytope, do Symington cuts on the plane (as in eigenvray diagrams)

Can equip any smooth manifold with eigenvray diagram type symplectic structure. Moreover, this structure is at least
conjecturally more meaningful from cluster variety viewpoint. (as above)

III) Relative symplectic cohomology.

\[ M \text{ geometrically bounded (e.g. } \mathbb{M}_K) \text{ symplectic manifold.} \]

1) \( \mathcal{K}\mathcal{M} \) compact set

\[ \mathcal{H}^*(\mathcal{K}, \mathcal{M}) \]

is an algebra over the Novikov field

\[ \Lambda := \{ \sum_{i=0}^{a_i} T^{b_i} | a_i \in \mathbb{K}, b_i \in \mathbb{R}, a_i \to \infty \} \]

It is defined using Hamiltonian Floer theory (Seidel, McLean, Venkatesh, Grimm, V.)
In this generality, this is a rather new invariant. One property to mention is the existence of restriction maps: $\text{St}^k_M(K) \rightarrow \text{St}^l_M(K)$.

2) If $U \subseteq M$ is an open set, then we define

$$\text{St}^k_M(U) = \lim_{K \downarrow 0} \text{St}^k_M(K)$$

where $K_1 \subset K_2 \subset \cdots$ is an exhaustion of $U$. (restriction maps $\check{\rightarrow}$)

Note the special case

Almost Then $\text{St}_M(M)$

A symplectomorphism invariant of $M$. 
3) Assume that $U \subset M$ is also geometrically bounded + dissipative (e.g. if $U$ is conical at $\infty$)

This assumption is satisfied, but that is not the only relevant case. It is enough for our purposes today.

A good picture to have in mind for such $U \subset M$:

$$U = M - \partial M$$
U is any U that is reachable from D under Liouville flow.

**Thm (Gromov-U.)** Let K ⊆ U be a compact subset. Then there is a canonical isomorphism.

\[ \text{SH}_U(K) \sim \text{SH}_M(K) \]

- Compatible with restriction maps.
- Relatively isos.

**Cor.** There is a canonical iso.

\[ \text{SH}_U(U) \sim \text{SH}_M(U) \]

We will apply these statements to the embeddings \( M_0 \to M_R \)
\( M_k \to M_R \) etc.
4) Let us do an example:

Let \( M_0 = T^* T^2 \)

\[
\begin{array}{c}
\xymatrix{\ar @{-} [d] \ \\ \ar @{-} [r] \ \\
B_0 = \mathbb{R}^2}
\end{array}
\]

Recall: Kontsevich - Soibelman sheaf on \( \mathbb{R}^2 \)

convex polygon \( \mathcal{F} \subset \mathbb{R}^2 \)

\[
KS(\mathcal{F}) = \left\{ \sum_{\alpha} x_1^{x_{1,\alpha}} x_2^{x_{2,\alpha}} \middle| \alpha \in \Lambda \right\}
\]

For every \( p \in \mathcal{F} \), \( \text{vol}(a_\alpha) - \alpha_1 p_1 - \alpha_2 p_2 \to \infty \)

"convergent formal power series"
Thm (Grothendieck) There exists a canonical isomorphism

\[ SH^0_{M_0}(\pi^{-1}(P)) \cong KS(P) \]

The way to prove this is to use \( V \)-shadows \( SH \)

\[ VSH^0(M_0) \otimes L \cong \Lambda \{ x_1^\alpha, x_2^\beta \} \]

and that both sides are certain completions with respect to a non-arch. norm defined by \( P \) that is preserved under the isomorphism.

Cor: \( SH^0_{M_0}(M_0) \) is isomorphic to the algebra of entire
functions on the non-archimedean torus \((\mathbb{C}^*)^2\).

6) Wall crossing

Consider

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

By locality we obtain an isomorphism.

\[ \text{Iso}_1 : H_{M_0}^0(\pi^*(P)) \cong \text{Sym}^0_{M_1}(\pi^*(P)) \]

But recall that I can do a branch move and obtain another embedding \(M_0 \hookrightarrow M_1\) which contains \(P\) in its image.
\( \sim \text{ iso}_2 : SH^0_{M_o}(\pi^*(P)) \sim SH^0_{M_1}(\pi^*(P)) \)

Using our previous computation, we obtain map

\( \text{iso}_1 \circ \text{iso}_2^{-1} : KS(P) \rightarrow KS(P) \)

**Thm (Gromov - V):** This map is given by

\[ z_1 \rightarrow z_1 \]

\[ z_2 \rightarrow z_2(1 + z_1) \]

What’s going on?

The locality isomorphisms are obtained by using almost complex structures that are tailored to take advantage of the geometric strong
boundedness of the given open subset.

So to get the two isomorphisms above, we need to use two different almost complex structures.

\[ \text{SH}_m(p, J_0) \xrightarrow{\text{cont.}} \text{SH}_m(p, J_1) \]

is

\[ \text{KS}(p) \quad \longrightarrow \quad \text{KS}(p) \]

It turns out that as \( J_0 \)
changes to \( J_1 \) slowly, there must come a time \( J_k \) which is not a regular \( J \). This introduces a non-trivial solution
for the continuum map. (This is intentionally vague)
We do not explicitly compute this but use an indirect approach that has an analytic flavor. It doesn't involve computing any actual Floer trajectories.

6) Given an eigenvary diagram $R$ we obtain $BR$.

Inside $BR$ we can talk about a $G$-topology.

This is "generated by" rational slope convex (makes sense) closed polygons, which do not contain a node at their boundary.
Note: Finding convex polytopes containing multiple notes is harder than one might imagine (may be impossible; see Mandel)

Using a Lagrangian fibration

\[ \text{we can define a sheaf on } B_k \]

(my thesis)
obtain a sheaf \( \mathcal{S}_2 \) on \( B_1 \)

\[ \mathcal{S}_2(\cdot) \]
Almost sheaves (Grauert-V.):

\[ \text{KS}_2(\cdot) \quad \text{and} \quad \text{SH}^0_M(\pi^{-1}(\cdot)) \]

are isomorphic sheaves.

Corollary:

- \( \text{SH}^0_{M_1}(M_1) \cong \text{entire functions on } \{ z(xy-1) = 1 \} \)

- \( \text{SH}^0_{M_1}(M_1) \quad \text{(cover } M_0, C^2, M_1) \)

\[ \downarrow \]

\( \text{SH}^0_{M_0}(M_0) \quad (\cong \text{SH}^0_{M_1}(M_0)) \)

is isomorphic to the application of the following factors to the inclusion of a \((C^2)^2\) chart into \(C^2 - \{ xy = 1 \}\)
1) Base change to \( \Lambda \)
2) Analytify
3) Take entire functions.

IV - Mirror symmetry.

1) \( \mathbf{R} \rightarrow M_k \) with its \( \mathbf{R} \)-adapted cover by \( M_i \)’s.

\[
\text{ith } M_i = V_i, \\
V_i \cap V_j = V_j \cap V_k = : W
\]

\[
\begin{array}{c}
\text{diagram} \\
\text{SH}^0_{M_i}(V_i) \downarrow \text{SH}^0_{M_k}(W) \\
\downarrow \text{SH}^0_{M_k}(V_k) \\
\downarrow \text{SH}^0_{M_k}(V_k) \\
\end{array}
\]

(assume all \( k_i = 1 \))
This diagram is isomorphic to the entire function (analytic) of the cluster structure.
We define \( Y^{\text{alg}}_R \) to be the colimit of \((b)\) (a scheme over \(\Lambda\)).

\( Y^{\text{alg}}_R \) is a "cluster variety over \(\Lambda\)." It is one incarnation of the B-side of \( \mathcal{M}_R \). Its global invariants are related to a Viterbo \( S^1 \) construction on \( \mathcal{M}_R \) but we have not worked out the details. Let us also note that for \( \lambda = 0 \), \( Y^{\text{alg}}_R \) seems to be a base change of \( Y_R \) (holomorphic vol. forms would match too).

2) We can also construct a non-archimedean \( \Lambda \) analytic space \( Y^\text{an}_R \) by gluing affinoid domains.
from the base (details omitted)

Very expected conjecture we are only on

- $Y_R$ admits a non-archimedean
  SYT fibration over $B_R$
- $Y_R$ is the analytification of $Y_{alg}^R$

There is a subtlety for dealing
with $k_i > 1$. The mirrors of local
models are not affine-affinoid. One
can either split up to simple ones or
 glue schemes/k-analytic spaces.
V - Homological mirror symmetry.

1) We hope to be able to prove similar locality theorems for Lagrangians that are well behaved at $\infty$.

Most important examples are sections of $\mathcal{M}_k \to \mathcal{B}_k$.

We can then do all of our constructions using $LH^*_H(K; \mathcal{P}_k)$. One can see this as we are constructing a $\mathcal{A}_k$-module over $\mathcal{Y}_k$, but it might better to just construct...
2) The key property of Ref is that it should locally generate the local Fukaya categories.

Thus, for any other compact Lagrangian, we would define the HMS functor as

\[ L \rightarrow LHM^*(\cdot ; \text{Ref}, L) \]

and expect it to be fully-faithful.

This part is ongoing work with Abozaid & Gromov.
If we are not interested in Fukaya category but just Lagrangian Floer theory of one Lagrangian, there are easier ways. It would be much easier to associate to a given Lagrangian a coherent sheaf over $\mathbb{R}$.

For the diagram:
- Tropical Lagrangian torus (disappears)
- $\mathbb{C}P^2$ elliptic curve
- The cluster variety described in GHK.
Remark: We also have some ideas about computing the "tropicalization" of the mirror map & closed string mirror symmetry.