The Poisson bracket invariant: Elementary and Hard Approaches

Shira Tanny

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PLAN:

• Background (Polterovich’s conjecture)

• Elementary approach in dimension 2
  Joint with Lev Buhovsky and Aleksandr Logunov

• Grothendieck’s theorem and linear symplectic geometry
  Joint with Efim Gluskin

• Floer homology of Hamiltonians supported on subsets
  Joint with Yaniv Ganor
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The Poisson bracket invariant

Recall:

\[
\{ f, g \} := \frac{d}{dt} f \circ \varphi^t g = \omega(X_f, X_g).
\]

\( U \subset M \) is displaceable if

\[ \exists f, \phi \quad f(U) \cap U = \emptyset. \]

Entov-Polterovich:

\[ (M; !) \text{ closed.} \]

Let

\[ \bullet U := \{ U_i \}_{i=1}^N \text{ an open cover of } M, \]

\[ \bullet F := \{ f_i \}_{i=1}^N \text{ a subordinate partition of unity.} \]

If the sets are displaceable, then there exist

\[ i; j \] with

\[ \{ f_i; f_j \} \neq 0 : \]

Definition

The Poisson bracket invariant is defined by

\[
\text{pb}(F) := \max \left| x_i \right|, \left| y_j \right| \leq \frac{1}{n} \sum x_i f_i \circ X_j y_j f_j \in C_0.
\]

\[
\text{pb}(U) := \inf F < U \text{ pb}(F).
\]
Recall: \( \{f, g\} := \frac{d}{dt} f \circ \varphi^t_f = \omega(X_f, X_g) \).

Entov-Polterovich
The Poisson bracket invariant

Recall: \( \{f, g\} := \frac{d}{dt} f \circ \phi_g^t = \omega(X_f, X_g) \).

\( U \subset M \) is displaceable if \( \exists f, \varphi_f^1(U) \cap U = \emptyset \).

Entov-Polterovich
The Poisson bracket invariant

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\( U \subset M \) is displaceable if \( \exists f, \varphi_f \) such that \( \varphi_f(U) \cap U = \emptyset \).

**Entov-Polterovich:** \((M, \omega)\) closed. Let

- \( \mathcal{U} := \{U_i\}_{i=1}^N \) an open cover of \( M \),
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If the sets are displaceable, then there exist \( i, j \) with \( \{f_i, f_j\} \neq 0 \):

- a cover by displaceable sets

**Definition**

The Poisson bracket invariant is defined by

\[
pb(F) := \max |x_i|, |y_j| \leq 1 \sum x_i f_i \wedge X_j y_j f_j \in C_0.
\]

\[
pb(U) := \inf F < U pb(F)
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The Poisson bracket invariant

Recall: \( \{f, g\} := \frac{d}{dt} f \circ \varphi_g^t = \omega(X_f, X_g) \).

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pb(\mathcal{F}) :=
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**Definition**
The Poisson bracket invariant is defined by

\[
\text{pb}(\mathcal{F}) := \left\{ \sum_i x_i f_i, \sum_j y_j f_j \right\}
\]

one strip is not displaceable
The Poisson bracket invariant

Recall: \( \{f, g\} := \frac{d}{dt} f \circ \varphi^t_g = \omega(X_f, X_g) \).

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\[
pb(\mathcal{F}) := \left\| \left\{ \sum_i x_if_i, \sum_j y_jf_j \right\} \right\|_{C^0},
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The Poisson bracket invariant

Recall: \( \{ f, g \} := \frac{d}{dt} f \circ \varphi_g^t = \omega(X_f, X_g) \).

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**Definition**
The Poisson bracket invariant is defined by

\[
pb(\mathcal{F}) := \max_{\sum |x_i|, |y_j| \leq 1} \left\| \sum_i x_i f_i, \sum_j y_j f_j \right\|_{C^0},
\]

one strip is not displaceable
The Poisson bracket invariant

Recall: \( \{ f, g \} := \frac{\partial}{\partial t} f \circ \varphi_t^g = \omega(X_f, X_g) \).

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**Definition**
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\[
pb(\mathcal{F}) := \max_{|x_i|, |y_j| \leq 1} \left\| \sum_i x_i f_i, \sum_j y_j f_j \right\|_{C^0},
\]

\[
pb(U) := \inf_{\mathcal{F} \subset U} pb(\mathcal{F}).
\]
Polterovich’s Conjecture:
There exists a constant $C = C(M, \omega)$ depending only on the symplectic manifold, such that

$$\text{pb}(\mathcal{U}) \cdot e(\mathcal{U}) \geq C$$
**The Poisson bracket invariant**

**Polterovich’s Conjecture:**
There exists a constant $C = C(M, \omega)$ depending only on the symplectic manifold, such that

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maximal

displacement energy
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There exists a constant $C = C(M, \omega)$ depending only on the symplectic manifold, such that

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maximal

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Related works:
Polterovich’s Conjecture:
There exists a constant $C = C(M, \omega)$ depending only on the symplectic manifold, such that

$$\mathcal{Pb}(U) \cdot \mathcal{E}(U) \geq C$$

↑

maximal

displacement energy

Related works:

Theorem (Buhovsky-Logunov-T.)
Let $(M, \omega)$ be a surface endowed with an area form. There exists a universal constant $C$ such that

$$\mathcal{Pb}(U) \cdot \mathcal{E}(U) \geq C.$$
Lemma (Buhovsky-Logunov-T.)
There exists a constant $c(n)$ depending only on the dimension, such that for every finite collection of functions $\mathcal{F} := \{f_i\}$,

\[
\frac{1}{c(n)} \left\| \sum_{i,j} |\{f_i, f_j\}| \right\| \leq \text{pb}(\mathcal{F}) \leq \left\| \sum_{i,j} |\{f_i, f_j\}| \right\|
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Lemma (Buhovsky-Logunov-T.)
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Originally, had $c(n) \propto \exp(n)$. 
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Originally, had $c(n) \propto \exp(n)$.

Using Grothendieck’s theorem from functional analysis:

Theorem (Gluskin-T.): $c(n) \leq 10\sqrt{n}$. 
Lemma (Buhovsky-Logunov-T.)
There exists a constant $c(n)$ depending only on the dimension, such that for every finite collection of functions $\mathcal{F} := \{f_i\},$

$$\frac{1}{c(n)} \left\| \sum_{i,j} \{f_i, f_j\} \right\| \leq pb(\mathcal{F}) \leq \Delta \leq \left\| \sum_{i,j} \{f_i, f_j\} \right\|$$

Pointwise inequality implies inequality for the norms

$$\frac{1}{c(n)} \sum_{i,j=1}^{N} |\omega(v_i, v_j)| \leq \max_{|x_i|, |y_j| \leq 1} \sum_{i,j} x_i \cdot y_j \cdot \omega(v_i, v_j)$$
Lemma (Buhovsky-Logunov-T.)
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Considering the matrix $A = (a_{ij})_{i,j=1}^{N}$ where $a_{ij} := \omega(v_i, v_j)$, this is equivalent to:
Lemma (Buhovsky-Logunov-T.)
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Theorem (Grothendieck / Pisier)

Let $A = (a_{ij})_{m \times n}$ be an $m \times n$ matrix. There exist:

- an $m \times n$ matrix $B$,
- vectors $(\mathbf{1}; \ldots; \mathbf{m})$ and $(\mathbf{1}; \ldots; \mathbf{n})$ with non-negative entries and Euclidean norms bounded by 1 such that

$$A = \text{diag}(\mathbf{1}; \ldots; \mathbf{m}) \cdot B \cdot \text{diag}(\mathbf{1}; \ldots; \mathbf{n})$$

and

$$\|B\|_{\ell^\infty_2, \ell^1_m} \leq K_G \cdot \|A\|_{\ell^\infty_\infty, \ell^\infty_2}$$

Here $K_G$ is a universal constant ("Grothendieck’s constant"), whose exact value is unknown.
Theorem (Grothendieck / Pisier)

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Grothendieck’s theorem about factorization

Theorem (Grothendieck / Pisier)

Let $A = (a_{ij})_{i=1,j=1}^{m,n}$ be an $m \times n$ matrix. There exist:

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Theorem (Grothendieck / Pisier)

Let $A = (a_{ij})_{i=1}^{m}{}_{j=1}^{n}$ be an $m \times n$ matrix. There exist:

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$$A = \text{diag}(\lambda_1, \ldots, \lambda_m) \cdot B \cdot \text{diag}(\mu_1, \ldots, \mu_n)$$

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$$\|B\|_{\mathcal{L}(\ell_2^n, \ell_2^m)} \leq K_G \cdot \|A\|_{\mathcal{L}(\ell_\infty^n, \ell_1^m)}$$
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Grothendieck’s theorem about factorization

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Polterovich’s conjecture in higher dimensions.
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Polterovich’s conjecture

Floer homology; spectral invariants; lower bounds for pb (for aspherical manifolds:)

The action functional \( A_f \) is defined on the space of contractible loops in \( M \).

• \( CF^* (f) \) is generated by \( \text{Crit}(A_f) = \) 1-periodic orbits of \( f \), \( (t) = t f(0) \).

• \( @: CF^*(f) \to CF^*_{-1}(f) \) counts negative gradient flow-lines of \( A_f = \) certain cylinders between periodic orbits.

\[ \Rightarrow HF^*(f) : = H^*(CF^*(f) ; @) \sim H^*(M) \].

Entov-Polterovich-Zapolsky: lower bounds in terms of spectral width of the sets.
Decay with the number of sets.

Polterovich: spectral width of disjoint unions of the sets.

\[ \Rightarrow \] Study spectral invariants of disjointly supported Hamiltonians:
Seyfaddini, Ishikawa, Humilière-Le Roux-Seyfaddini

Theorem (HLS, “Max-formula”)

Suppose \( F \) and \( G \) are supported in disjoint incompressible Liouville domains on a symplectically aspherical manifold. Then,

\[ c(F + G ; [M]) = \max \{ c(F ; [M]) , c(G ; [M]) \} \]
Polterovich’s conjecture in higher dimensions

- Floer homology
- spectral invariants
- lower bounds for $pb$

The action functional $A_f$ is defined on the space of contractible loops in $M$.

- $\text{CF}^*(f)$ is generated by $\text{Crit}(A_f) = 1$-periodic orbits of $f$, $(t) = \frac{d}{dt} f(t)$.
- $\partial: \text{CF}^*(f) \to \text{CF}^{* - 1}(f)$ counts negative gradient flow-lines of $A_f = \text{certain cylinders between periodic orbits}$.  

$HF^*(f) := H^*(\text{CF}^*(f))$; $\partial \sim H^*(M)$.

Entov-Polterovich-Zapolsky: lower bounds in terms of spectral width of the sets. Decay with the number of sets.

Polterovich: spectral width of disjoint unions of the sets.

$\Downarrow$

Study spectral invariants of disjointly supported Hamiltonians: Seyfaddini, Ishikawa, Humilière-Le Roux-Seyfaddini

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Polterovich’s conjecture in higher dimensions

Floer homology $\sim$ spectral invariants $\sim$ lower bounds for $pb$

(for aspherical manifolds:) The action functional $\mathcal{A}_f$ is defined on the space of contractible loops in $M$. 
Polterovich’s conjecture in higher dimensions

**Floer homology** $\sim$ **spectral invariants** $\sim$ **lower bounds for pb**

(for aspherical manifolds:) The action functional $\mathcal{A}_f$ is defined on the space of contractible loops in $M$.

- $\text{CF}_*(f)$ is generated by $\text{Crit}(\mathcal{A}_f) = 1$-periodic orbits of $f$, $\gamma(t) = \varphi_f^t(\gamma(0))$. 

$\mathcal{L}M$
Polterovich’s conjecture in higher dimensions

Floer homology $\sim$ spectral invariants $\sim$ lower bounds for $pb$

(for aspherical manifolds:) The action functional $\mathcal{A}_f$ is defined on the space of contractible loops in $M$.

- $CF_\ast(f)$ is generated by $\text{Crit}(\mathcal{A}_f) = 1$-periodic orbits of $f$, $\gamma(t) = \varphi_f^t(\gamma(0))$.

- $\partial : CF_\ast(f) \to CF_{\ast-1}(f)$ counts negative gradient flow-lines of $\mathcal{A}_f = \text{certain cylinders between periodic orbits.}$
Polterovich’s conjecture in higher dimensions

Floer homology $\rightsquigarrow$ spectral invariants $\rightsquigarrow$ lower bounds for $pb$

(for aspherical manifolds:) The action functional $A_f$ is defined on the space of contractible loops in $M$.

- $CF_*(f)$ is generated by $Crit(A_f) = 1$-periodic orbits of $f$, $\gamma(t) = \varphi_f^t(\gamma(0))$.

- $\partial: CF_*(f) \to CF_{*-1}(f)$ counts negative gradient flow-lines of $A_f = \text{certain cylinders between periodic orbits}$. 

$\Rightarrow HF_*(f) := H_*(CF_*(f), \partial) \cong H_*(M)$. 

Entov-Polterovich-Zapolsky: lower bounds in terms of spectral width of the sets. 

Decay with the number of sets. 

Polterovich: spectral width of disjoint unions of the sets. 

⇓ Study spectral invariants of disjointly supported Hamiltonians: Seyfaddini, Ishikawa, Humilière-Le Roux-Seyfaddini 

Theorem (HLS, “Max-formula”) 

Suppose $F$ and $G$ are supported in disjoint incompressible Liouville domains on a symplectically aspherical manifold. Then, 

$$c(F + G; [M]) = \max\{c(F; [M]), c(G; [M])\}.$$
Polterovich’s conjecture in higher dimensions

Floer homology \sim \text{spectral invariants} \sim \text{lower bounds for } pb
Polterovich’s conjecture in higher dimensions

Floer homology $\mapsto$ spectral invariants $\mapsto$ lower bounds for $pb$

**Improper Definition:** For $\alpha \in H_*(M) \cong HF_*(f)$, the spectral invariant $c(f; \alpha)$ is the smallest action of a representative of $\alpha$ in $CF_*(f)$. 
Polterovich’s conjecture in higher dimensions

Floer homology $\sim$ spectral invariants $\sim$ lower bounds for $pb$

**Improper Definition:** For $\alpha \in H_*(M) \cong HF_*(f)$, the spectral invariant $c(f; \alpha)$ is the smallest action of a representative of $\alpha$ in $CF_*(f)$.

**Example:**

![Diagram](image)
Polterovich’s conjecture in higher dimensions

Floer homology \rightarrow \text{spectral invariants} \rightarrow \text{lower bounds for } \operatorname{pb}

**Improper Definition:** For \( \alpha \in H_\ast(M) \cong HF_\ast(f) \), the spectral invariant \( c(f; \alpha) \) is the smallest action of a representative of \( \alpha \) in \( CF_\ast(f) \).

**Example:**

\[ c(f, [pt]) = \min f \]
Polterovich’s conjecture in higher dimensions

Floer homology $\sim$ spectral invariants $\sim$ lower bounds for $pb$

**Improper Definition:** For $\alpha \in H_\ast(M) \cong HF_\ast(f)$, the spectral invariant $c(f; \alpha)$ is the smallest action of a representative of $\alpha$ in $CF_\ast(f)$.

Example:

- $c(f; [\gamma]) = f(x)$
- $c(f, [pt]) = \min f$
Polterovich’s conjecture in higher dimensions

Floer homology \sim \rightarrow \text{spectral invariants} \sim \rightarrow \text{lower bounds for } pb

**Improper Definition:** For \( \alpha \in H_\ast(M) \cong HF_\ast(f) \), the spectral invariant \( c(f; \alpha) \) is the smallest action of a representative of \( \alpha \) in \( CF_\ast(f) \).

**Example:**

- \( c(f; [M]) = \max f \)
- \( c(f; [\gamma]) = f(x) \)
- \( c(f, [pt]) = \min f \)
Polterovich’s conjecture in higher dimensions

Floer homology $\leadsto$ spectral invariants $\leadsto$ lower bounds for $pb$

Entov-Polterovich-Zapolsky: lower bounds in terms of spectral width of the sets.
Polterovich’s conjecture in higher dimensions

Floer homology \xrightarrow{\sim} \text{spectral invariants} \xrightarrow{\sim} \text{lower bounds for pb}

\textbf{Entov-Polterovich-Zapolsky}: lower bounds in terms of spectral width of the sets. Decay with the number of sets.
Polterovich’s conjecture in higher dimensions

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⇓

Study spectral invariants of disjointly supported Hamiltonians:
Polterovich’s conjecture in higher dimensions

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Study spectral invariants of disjointly supported Hamiltonians: Seyfaddini, Ishikawa, Humilière-Le Roux-Seyfaddini
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\[ \Downarrow \]

Study spectral invariants of disjointly supported Hamiltonians: Seyfaddini, Ishikawa, Humilière-Le Roux-Seyfaddini

Theorem (HLS, ”Max-formula”)
Suppose \( F \) and \( G \) are supported in disjoint incompressible Liouville domains on a symplectically aspherical manifold. Then,

\[ c(F + G; [M]) = \max\{c(F; [M]), c(G; [M])\}. \]
Floer theory of Hamiltonians supported in subsets.
Assume that:

- \((M, \omega)\) is closed, symplectically aspherical \((\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0)\).
Setting

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- The Hamiltonians are supported in disjoint embeddings of "nice" star-shaped domains.
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More generally, consider domains with contact-type, incompressible boundaries. Call these CIB domains.
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More generally, consider domains with contact-type, incompressible boundaries. Call these CIB domains.

\[ i_*: \pi_1(\partial U) \to \pi_1(M) \] injective
Setting

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- The Hamiltonians are supported in disjoint embeddings of "nice" star-shaped domains.

More generally, consider domains with contact-type, incompressible boundaries. Call these CIB domains.

ok for us:  

not ok:
Theorem (Ganor-T.):
• $(M_i; \Omega_i)$ are symplectically aspherical,
• $V$ is compact with contact-type boundary,
• $i: V, \rightarrow M_1$, $j: V, \rightarrow M_2$, images are CIB domains.

For every $F$ supported in $V$:
$$c(i^* F; [M_1]) = c(j^* F; [M_2]).$$

Remark:
The asphericity and incompressibility assumptions are necessary.
Theorem (Ganor-T.):

- $(\Omega_i; \sigma_i)$ are symplectically aspherical,
- $V$ is compact with contact-type boundary,
- $\iota: V, \rightarrow M_1, \eta: V, \rightarrow M_2$, images are CIB domains.

For every $F$ supported in $V$:

$$c_{M_1}(\iota_\ast [M_1]) = c_{M_2}(\eta_\ast [M_2]).$$

Remark:
The asphericity and incompressibility assumptions are necessary.
Theorem (Ganor-T.):

- \((M_i; V)\) are symplectically aspherical,
- \(V\) is compact with contact-type boundary,
- \(i: V, j: V \to M_1, j: V \to M_2\) images are CIB domains.

For every \(F\) supported in \(V\):

\[ c(i_*F; [M_1]) = c(j_*F; [M_2]) \]

Remark: The asphericity and incompressibility assumptions are necessary.
Results

Theorem (Ganor-T.):

- \((M_i; q_i)\) are symplectically aspherical,
- \(V\) is compact with contact-type boundary,
- \(i: V \to M_1, j: V \to M_2\) images are CIB domains.

For every \(F\) supported in \(V\):

\[ c(i_*F; [M_1]) = c(j_*F; [M_2]) \]

Remark: The asphericity and incompressibility assumptions are necessary.
Theorem (Ganor-T.):

- \((M_i, \omega_i)\) are symplectically aspherical,
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- \(i : V \hookrightarrow M_1, j : V \hookrightarrow M_2\), images are CIIB domains.

For every \(F\) supported in \(V\):

\[
c(i_* F ; [M_1]) = c(j_* F ; [M_2]).
\]
**Results**

**Theorem (Ganor-T.):**

- \((M_i, \omega_i)\) are symplectically aspherical,
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For every \(F\) supported in \(V\):  
\[
c(i \ast F; [M_1]) = c(j \ast F; [M_2]).
\]

**Remark:**  
The asphericity and incompressibility assumptions are necessary.
Theorem (Ganor-T.):
Let $F$ and $G$ be Hamiltonians supported in disjoint CIB domains, then:

1. $c(F + G; \cdot) \leq \max\{c(F; \cdot), c(G; \cdot)\}$, for every $H^* \in H^*(M)$.

2. $(F + G) \geq \max\{F, G\}$, where $\cdot$ is the boundary depth.

3. $c_{AHS}(F + G) \leq \min\{c_{AHS}(F), c_{AHS}(G)\}$, where $c_{AHS}$ is an action selector defined recently by Abbondandolo, Haug and Schlenk.

Definition:
For a non-degenerate Hamiltonian $F$, consider homotopies of Hamiltonians $H$ and $J$ such that $H - J = F$, and denote by $M(H, J)$ is the set of solutions of Floer equation with respect to $(H, J)$.
Then, $c_{AHS}(F) = \sup (H, J) \min_{u \in M(H, J)} A_F(u(-\infty))$. 

Theorem (HLS):
On surfaces (other than $S^2$) and for autonomous Hamiltonians, every action selector satisfying the min-formula coincides with $c(\cdot; \cdot)$. 
Theorem (Ganor-T.): Let $F$ and $G$ be Hamiltonians supported in disjoint CIB domains, then:

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Definition: For a non-degenerate Hamiltonian $F$, consider homotopies of Hamiltonians $H$ and a.c.s. $J$ such that $H - F = 0$, and denote by $M(H, J)$ is the set of solutions of Floer equation with respect to $(H, J)$. Then, $c_{AHS}(F) = \sup_{(H, J)} \min_{u \in M(H, J)} A_F(u(\infty))$. 

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Theorem (HLS):
On surfaces (other than $S^2$) and for autonomous Hamiltonians, every action selector satisfying the min-formula coincides with $c(\cdot; [pt])$. 
Locality in Morse homology

1. Lines starting in $V$, away from the boundary, are contained in $V$.
2. Lines ending in $V$ are contained in $V$.

Starting on the "bump", can flow both in and out.
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Starting on the “bump”, can flow both in and out.
The main tool: Barricades

**Theorem (Ganor-T.):**
Suppose $F$ is supported in a CIB domain $V$. Then, there exists a perturbation $f$ of $F$, and an almost complex structure $J$, such that for every solution $u$ of the Floer equation with respect to $(f, J)$:

1. If $u$ starts in $V \setminus \mathcal{N}(\partial V)$, then $\text{im}(u) \subset V \setminus \mathcal{N}(\partial V)$.
2. If $u$ ends in $V$, then $\text{im}(u) \subset V$. 
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[Diagrams showing allowed and forbidden trajectories]
The main tool: Barricades

Under the decomposition

$$CF(f) = C_{V \setminus \mathcal{N}(\partial V)} \oplus C_{M \setminus V} \oplus C_{\mathcal{N}(\partial V)},$$

the differential takes a triangular block form:

$$\partial_{f,j} = \begin{pmatrix}
\partial|_{V \setminus \mathcal{N}(\partial V)} & 0 & \partial|_V \\
0 & * & * \\
0 & 0 & \partial|_V
\end{pmatrix}$$
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0 & * & *
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\]
Constructing Barricades:

\[ \Gamma := u^{-1}(\partial V) \]

Claim: \( A(\Gamma) > \max_{V_c} f \)

\( + \) slope

if non-aspherical

contradict to action decr. along \( u \).

incomp: \( \int_{\partial U} \omega = \int_{\Gamma} \alpha \)
"Proof" of locality:

\( \text{Supp}(F) \subset V \)

\[ i(V) \]

\[ M_1 \]

\[ M_2 \]

Want:  \[ \gamma_{M_1}^u(x; [f]) = \gamma_{M_2}^u(x; [f]) \]

Pert w. Barricades

\[ \alpha \in CF(f) \text{ rep } [f] \]

of lowest action

Assume: \( \alpha \subset V \setminus W(AV) \)

\[ \exists \alpha' = 0 \]

Barrière

| (1) \( \tilde{f} \alpha' = 0 \) |
| (2) \( \alpha' | \text{im}(\tilde{f}) \) |

Resolved using continuation maps corresponding to homotopies \( f, f' \to \text{small Morse} \).
Thank you!