May 23

**Gluing.** Consider the setting of Floer theory. Let \( \dim \mathcal{M}(x, y) = \dim \mathcal{M}(y, z) = 1 \). Then there exists a gluing map

\[
\text{glue} : \hat{\mathcal{M}}(x, y) \times (R_0, \infty) \times \hat{\mathcal{M}}(y, z) \to \hat{\mathcal{M}}(x, z)
\]

which satisfies the following property: as \( R \to \infty \),

\[
\text{glue}(u, v, R) \to u \cup v \text{ in the Gromov sense.}
\]

The idea is basic: “glue” \( u \in \mathcal{M}(x, y) \) and \( v \in \mathcal{M}(y, z) \) to get \( Z \to M \) which is close to being in \( \mathcal{M}(x, z) \). One thing to keep in mind is that everything should be done parametrically in the gluing parameter \( R \).

We define the pre-gluing of two curves \( u, v \) by the following picture

\[
\begin{array}{ccc}
  & v(s + R, t) & \\
-u \#_R v = & y(t) & u(s - R, t) \\
-\frac{R}{2} & & \frac{R}{2}
\end{array}
\]

where we use the Riemannian exponential function to interpolate between the two solutions. This is well-defined for \( R \) sufficiently large since \( u(t, s) \to y(t) \) as \( s \to \infty \) (and similarly for \( v(t, s) \)). We will start using only \( u \) for \( s > 3R/4 \) and only \( v \) for \( s < -3R/4 \). Interpolation only happens when \( |s| \) is in \([R/2, 3R/4]\). As \( R \to \infty \), the left side covers all of \( v \) and the right side covers all of \( u \).

It is important to note that the pregluing \( u \#_R v \) is not actually a solution of the Floer equation. However, one can show that a linearization of \( \partial \) at \( u \#_R v \) is surjective (assuming that the moduli space \( \mathcal{M}(x, z) \) is cut transversally). You can use the implicit function theorem to show that there is an actual solution nearby. The hard part is to make sure that for every \( R \) you get a different solution. We want this because we want the the glueing map to be a diffeomorphism onto a “boundary” neighborhood of \( u \cup v \).

**Newton’s iteration.** The way one turns the pregluing map into a gluing map is based on Newton’s iteration to find zeros of a function. Recall:

**Proposition 1.** Let \( f : \mathbb{R}^k \to \mathbb{R}^k \) be a smooth map and suppose \( f(x_0) = 0 \) and \( df_{x_0} \) is invertible. Define a map

\[
x \in \mathbb{R}^k \mapsto \varphi(x) := x - df_{x_0}^{-1} \cdot f(x)
\]

Then \( \varphi \) is a contraction on a neighborhood of \( x_0 \).

**Proof.** Compute

\[
df_{x_0}(\varphi(x) - \varphi(y)) = f(x) - f(y) - df_{x_0}(x - y).
\]
By continuous differentiability of $f$, we may suppose that there is a small neighborhood around $x_0$ so that for all $x, y$ in this neighborhood we have
\[
|f(x) - f(y) - df_{x_0}(x - y)| \leq \frac{1}{2} |df_{x_0}|^{-1} |x - y|,
\]
whence $|\varphi(x) - \varphi(y)| < 1/2|x - y|$, as desired. \(\square\)

Let’s first explain how we hope to apply Newton’s method to our problem. Consider the embedding $R \mapsto u \# R v$. We can think of this as defining an embedded curve $\Sigma$ in the ambient space of maps. The moduli space $M(x, z)$ is the zero set of a section. Newton’s method prescribes a way to converge to the zero set of a function, and hopefully it can be applied parametrically to “project” $\Sigma$ onto $M(x, z)$ (in a one-to-one fashion) – this will be how we turn our pre-gluing map into a gluing map.

Let’s look at a simpler situation where we hope to apply Newton’s method parametrically. Consider a map $f : R^2 \to R$. Let $\Sigma$ be an embedded one-parameter family of points in $R^2$, which is “close” in some sense to $f^{-1}(0)$. Let $p$ be a point on $f^{-1}(0)$, and suppose that $df_p : R^2 \to R$ is surjective. The naive generalization of Newton’s method to the parametric setting would be pick a right inverse $L_p : R \to R^2$ for $df_p$ and to consider the map
\[
\varphi(x) = x - L_p f(x).
\]

The observation is that fixed points of $\varphi$ are precisely the points on the zero locus. Unfortunately one quickly realizes that $\varphi$ cannot be a contraction, since it has multiple fixed points in arbitrarily small neighborhoods of $p$. Therefore, it is not obvious that the Newton iteration $x, \varphi(x), \varphi^2(x), \cdots$ converges, and so a priori we do not get a “projection” onto the zero locus.

**Idea.** The idea is to foliate a neighborhood of $\Sigma$ so that the map $f$ is bijective on the leaves, and then show that Newton’s iteration converges on each leaf. We should think of leaves as obtained by exponentiating the subspaces given by the inverse $L$.

We still need to construct the right-inverses (subspaces) with some uniform upper bounds (on the derivative of the chosen right inverse of $df$) and Salomon does that in the notes.

Henceforth we will assume that we have defined an open embedding
\[
\text{glue} : \hat{M}(x, y) \times (R_0, \infty) \times \hat{M}(y, z) \to \hat{M}(x, z),
\]
whenever $\hat{M}(x, y)$ and $\hat{M}(y, z)$ are 0-dimensional and $\hat{M}(x, z)$ is 1-dimensional, which satisfies the aforementioned convergence to the broken flow lines as $R \to \infty$. These gluing maps give us charts for the moduli space near the boundary points – we conclude that $\hat{M}(x, z)$ can be compactified into a one-dimensional manifold with boundary.

**Orientations.** We need to have $\partial^2 = 0$ which immediately follows from compactness and gluing when we work over $\mathbf{F}_2$ (or any algebra over $\mathbf{F}_2$).

\[
\langle \partial^2 x, z \rangle = \sum_y \#M(x, y) \times_y \hat{M}(y, z) = \# \partial M(x, z).
\]
This argument about why $\partial^2 = 0$ is originally due to Floer.

If we are not working over $\mathbf{F}^2$, then we need to assign $\pm 1$ to each rigid solution (i.e. index 1) so that $\partial^2 = 0$.

**The Determinant bundle.** It is important to notice that $\Lambda^0$ (any $\mathbb{R}$-vector space) $= \mathbb{R}$ (including the zero vector space). An orientation of a manifold is an orientation of the real-line bundle $\Lambda^{\text{max}} TM$.

Let $V, W$ be $\mathbb{R}$-vector spaces, let $\varphi : V \to W$ be a linear map. Define

$$\det(\varphi) = \Lambda^{\text{max}} \ker \varphi \otimes \Lambda^{\text{max}} (\text{coker} \varphi)^*.$$ 

If $X$ is a topological space, and $\Phi : X \to \text{Hom}(V, W)$ is continuous, then there is a canonical line bundle $\det \Phi \to X$ whose fiber at $x$ is identified with the vector space $\det \Phi(x)$.

There are similar statements for Fredholm operators.

**Example 2.** In our construction we have a Banach space bundle $E \to \mathcal{P}$ and a section $s = \bar{\partial}_{H,J}$. The derivative of the linearization (well-defined on the zero section) can be thought of as a continuous map $s^{-1}(0) \to \text{Fredholm Operators}$. We therefore obtain a determinant line bundle $\det(Ds) \to s^{-1}(0)$. If $s$ is transverse to the zero section (i.e. if $H, J$ are chosen regular), then

$$\det(Ds) = \Lambda^{\text{max}} \ker(Ds) = \Lambda^{\text{max}} T s^{-1}(0) = \Lambda^{\text{max}} TM(x^\pm).$$

Therefore, orienting $\det Ds$ is equivalent to orienting the (parametrized) moduli space.

Moreover, if $u$ is rigid, $\mathcal{M}(x^\pm)$ has a canonical orientation at $u$ (obtained by translating the curve, i.e. moving in the direction $\partial_u u \in TM$).

Therefore we can orient the rigid solutions in two different ways, and therefore get signs according to whether the orientations agree or not.

**Families of linear operators.** In the next lecture, we will consider all linear operators of the form

$$\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + S(s, t) \xi,$$

where $S(s, t)$ are appropriately asymptotically constant at infinity.